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HIGHER K-THEORY FOR REGULAR SCHEMES

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ABSTRACT. Higher K-groups are defined for regular schemes, generalizing the K-theory of Karoubi and Villamayor. A spectral sequence is developed which shows how the K-groups depend on the local rings of the scheme. Applications to curves and affine surfaces are given.

Let X be a regular separated scheme. If U is an affine open subset of X, then the assignment $U \mapsto \mathrm{BGl}(S^n\Gamma(U, O_X)_*)$ is a sheaf of Kan complexes on the Zariski site. Here S denotes the suspension ring functor of Karoubi [**10**] and if A is a ring, A_* denotes the simplicial ring [**11**]

$$(A_*)_n = A[t_0, t_1, \ldots, t_n]/(t_0 + \cdots + t_n - 1).$$

We recall that $\pi_i \mathrm{BGl} A_* = K^{-i}A$, $i \geqq 1$ [**11**], where the K-groups of Karoubi and Villamayor are indicated [**10**]. Also, recall that $K_0(A) \times \mathrm{BGl}(A_*) \simeq \Omega\mathrm{BGl}(SA_*)$ if A is K-regular ([**9**], [**8**]). Thus there is a sheaf of Kan spectra $E(O_X)$ on X associated to the pre-spectrum $U \mapsto (n \mapsto \mathrm{BGl}(S^n\Gamma(U, O_X)_*))$. Such sheaves have been studied by K. Brown [**4**] who has defined cohomology with coefficients in a sheaf of Kan spectra: $H^n(X, E(O_X))$, $n \in Z$.

DEFINITION. $K^n(X) = H^n(X, E(O_X))$.

We remark that the spectra $E(O_X)$ are connected since X is regular, so $K^i(X) = 0$ if $i > 0$. The main properties of these groups and most of the motivation for introducing them are summarized in

THEOREM 1. *Let X be a regular separated scheme.*

(1) *If U and V are open subschemes of X, then there is an exact Mayer-Vietoris sequence*

$$\cdots \to K^{i-1}(U \cap V) \to K^i(U \cup V) \to K^i(U) \oplus K^i(V) \to K^i(U \cap V) \to \cdots$$

(2) *If X has finite (Krull) dimension, then there is a fourth quadrant spectral sequence of cohomological type*

$$E_2^{pq} = H^p(X, \underline{K}^q) \Rightarrow K^{p+q}(X).$$

Here $\underline{K}^q$ is the sheaf in the Zariski site associated to the presheaf

$$U \mapsto K^q(\Gamma(U, O_X)), \qquad U \textit{ affine open.}$$

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