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AN INVERSION FORMULA INVOLVING PARTITIONS

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In this note we outline a combinatorial proof of an inversion formula involving partitions of a number. This formula can be used to obtain the theory of symmetric group characters in a purely combinatorial way, as will be done in a forthcoming book, *The combinatorics of the symmetric group*, by the present author and Dr. G.-C. Rota.

The terminology we use is as follows. By a composition α of an integer n we mean a sequence $(\alpha_1, \alpha_2, \ldots, \alpha_s)$ of nonnegative integers whose sum is n. A partition of n is a composition $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_p)$ with $\lambda_1 \geqq \lambda_2 \geqq \cdots \geqq \lambda_p > 0$. The notation $\lambda \vdash n$ means "λ is a partition of n". We use the symbols α, β for compositions, λ, μ, ρ for partitions.

A Young diagram of shape λ is an array of dots, with λ_1 dots in the first row, λ_2 in the second row, etc., in which the first dots from the rows lie in a column, the second dots form a column, and so on. The conjugate partition $\tilde{\lambda}$ of λ is the shape obtained when the Young diagram of shape λ is transposed about its main diagonal, i.e., the rows of the transposed diagram are the columns of the original diagram. A generalized Young tableau (GYT) π of shape λ is an array of integers q_{ij} $(i = 1, 2, \ldots, p$, $j = 1, 2, \ldots, \lambda_i)$ with $q_{ij} > 0, q_{i,j+1} \geqq q_{ij}$ if $j < \lambda_i$, and $q_{i+1,j} > q_{ij}$ if $j \leqq \lambda_{i+1}$, i.e., an array of positive integers of shape λ which is increasing nonstrictly along the rows and increasing strictly down the columns. The type of a GYT π is the composition $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_s)$ of n (where $\lambda \vdash n$), where α_i is the number of times the integer i appears in π.

If $\alpha = (\alpha_1, \ldots, \alpha_s)$ is a composition of n with $s \leqq n$, and $\tau \in S_n$ (the symmetric group on $\{1, 2, \ldots, n\}$), then $\tau \cdot \alpha$ is the composition of n whose parts are $\alpha_i + \tau(i) - i$, $i = 1, 2, \ldots, n$ (where $\alpha_i = 0$ if $i > s$), if all these parts are nonnegative, and $\tau \cdot \alpha$ is undefined otherwise. We also define $\tau * \lambda$ to be the partition of n whose parts are $\lambda_i + \tau(i) - i$ in nonincreasing order if all these parts are nonnegative, and $\tau * \lambda$ is undefined otherwise.

Our inversion formula can now be stated.

THEOREM. *Let f, g be mappings from $\{\lambda | \lambda \vdash n\}$ to some field F of characteristic* 0. *Then*

$$f(\lambda) = \sum_{\tau \in S_n} (\operatorname{sign} \tau) g(\tau * \lambda) \leftrightarrow g(\lambda) = \sum_{\mu \vdash n} K_{\mu\lambda} f(\mu),$$

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