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THE REPRESENTATION OF LATTICES BY MODULES

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1. A quasivariety characterization of lattices representable by Λ-modules. If Λ is a nontrivial ring with 1, a lattice L is "representable by Λ-modules" if it can be embedded in the lattice of submodules of some unitary left Λ-module M. This lattice of submodules is denoted $\Gamma(M;\Lambda)$.

A (lattice) "Horn formula" is an open formula:

$$(e_1 = e_2 \,\&\, e_3 = e_4 \,\&\, \ldots \,\&\, e_{n-3} = e_{n-2}) \Rightarrow e_{n-1} = e_n,$$

where $e_1, e_2, \ldots, e_n$ are lattice polynomials.

MAIN THEOREM. *For every commutative ring Λ, there exists a set $J(\Lambda)$ of Horn formulas such that a lattice L is representable by Λ-modules if and only if every formula of $J(\Lambda)$ is satisfied in L. Each member of $J(\Lambda)$ is constructible by a finite sequence of four basic operations.*

That is, the class $\mathscr{L}(\Lambda)$ of lattices representable by Λ-modules is the "quasivariety" of lattices satisfying J(Λ), for commutative Λ.

OUTLINE OF PROOF. For Λ commutative, let $\iota: L \to \Gamma(M;\Lambda)$ be an embedding for some M. Without loss of generality, assume that L has a smallest element ω, and $\iota(\omega) = 0$. Motivated by the "abelian" lattice $\Gamma_f(G^N)$ of [**2**, 4.2] with $G = M$, we consider "constraint systems" in variables a_k (corresponding to coordinate positions in $M^{\mathbf{N}}$) and "auxiliary" variables b_k (with existential quantifiers understood) for k in $N = \{1, 2, 3, \ldots\}$. Consider $\boldsymbol{r} = (d_1, d_2, d_3, d_4)$ below.

(d_1) $\quad a_1 \in x_1, \quad a_2 \in x_2, \quad a_k \in \omega \quad$ for $\quad k \geqq 3 \; (x_1, x_2 \in L)$.

(d_2) $\quad b_1 \in x_3, \quad b_2 \in x_1, \quad b_k \in \omega \quad$ for $\quad k \geqq 3 \; (x_3 \in L)$.

(d_3) $$a_1 - a_2 - b_1 = 0.$$

(d_4) $$a_1 - \lambda_0 b_2 = 0 \qquad (\lambda_0 \in \Lambda).$$

A "solution" $f: \boldsymbol{N} \to M$ of $\boldsymbol{r}$ satisfies

(e_1) $\quad f(1) \in \iota(x_1), \quad f(2) \in \iota(x_2), \quad f(k) \in \iota(\omega) = 0 \quad$ for $k \geqq 3 \; (d_1)$.

(e_2) $$f(1) - f(2) \in \iota(x_3) \qquad (d_3, b_1 \in x_3).$$

(e_3) $\quad$ There exists $v \in \iota(x_1)$ such that $\lambda_0 v = f(1) \; (d_4, b_2 \in x_1)$.

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