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PALEY-WIENER THEOREMS AND SURJECTIVITY OF INVARIANT DIFFERENTIAL OPERATORS ON SYMMETRIC SPACES AND LIE GROUPS

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1. **Introduction.** The principal result of this paper is that if D is an invariant differential operator on a symmetric space X of the noncompact type then, for each function $f \in C^\infty(X)$, the differential equation $Du = f$ has a solution $u \in C^\infty(X)$. This is proved by means of a Paley-Wiener type theorem for the Radon transform on X. As a consequence we also obtain a Paley-Wiener theorem for the Fourier transform on X, that is an intrinsic characterization of the Fourier transforms of the functions in $C_c^\infty(X)$. In [**2**], Eguchi and Okamoto characterized the Fourier transforms of the Schwartz space on X. Invoking in addition the division theorem of Hörmander [**16**] and Lojasiewicz [**18**] we obtain by the method of [**11**] the surjectivity of D on the space of tempered distributions on X.

Finally, as a consequence of a structure theorem of Harish-Chandra [**8**] for the bi-invariant differential operators on a noncompact semisimple Lie group G, we obtain a local solvability theorem for each such operator.

2. **The range of invariant differential operators.** Let X be a symmetric space of the noncompact type, that is a coset space G/K where G is a connected, noncompact semisimple Lie group with finite center and K a maximal compact subgroup. Let $\boldsymbol{D}(X)$ denote the set of differential operators on X, invariant under G and let $C^\infty(X)$ denote the set of all C^∞ functions on X and $C_c^\infty(X)$ the set of $f \in C^\infty(X)$ of compact support.

THEOREM 2.1. *Let $D \neq 0$ in $\boldsymbol{D}(X)$. Then*

$$DC^\infty(X) = C^\infty(X).$$

As in Malgrange's proof of an analogous theorem for constant coefficient operators on $\boldsymbol{R}^n$ ([**3**], [**20**]) our proof proceeds by proving that if V is a closed ball in X then

$$f \in C_c^\infty(X), \operatorname{supp}(Df) \subset V \quad \textit{implies} \quad \operatorname{supp}(f) \subset V,$$

supp denoting support. This is proved by means of Theorem 2.2 below

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