BULLETIN OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 79, Number 1, January 1973

SOME VARIATIONAL PROBLEMS ON CERTAIN SOBOLEV SPACES AND PERFECT SPLINES

BY SAMUEL KARLIN

Communicated by R. Creighton Buck, May 8, 1972

The Sobolev space $W_\infty^{(n)}[a, b]$ is comprised of all functions f defined on $[a, b]$ where $f^{(n-1)}$ is absolutely continuous and the maximum norm of its nth derivative $\|f^{(n)}\|_\infty$ is finite. For ease of exposition, henceforth let $[a, b] = [0, 1]$. Let $x_1 < x_2 < \ldots < x_{n+r}$ be $n + r$ prescribed points in $[0, 1]$ and $\{\alpha_i\}_{i=1}^{n+r}$ given real numbers. Consider the subset $\mathscr{I}[\boldsymbol{x}, \boldsymbol{\alpha}]$ of $W_\infty^{(n)}$ consisting of all $f \in W_\infty^{(n)}$ interpolating the data $\{\alpha_i\}$ at $\{x_i\}$; i.e., $f \in W_\infty^{(n)}$ satisfies

$$f(x_i) = \alpha_i, \qquad i = 1, 2, \ldots, n + r. \tag{1}$$

A problem of interest in approximation theory is to characterize the function yielding

$$\min_{f \in \mathscr{I}[x, \alpha]} \|f^{(n)}\|_\infty \tag{2}$$

(for background on the problem see, e.g., Glaeser [**1**], Tihomirov [**5**], Schoenberg [**3**]). We will prove that the minimum is attained by a "perfect spline." A *perfect spline* of degree n with $r - 1$ knots in $[0, 1]$ is a spline polynomial of the special form

$$P(x) = c\left[x^n + 2\sum_{i=1}^{r-1} (-1)^i (x - \xi_i)_+^n\right] + \sum_{i=0}^{n-1} a_i x^i \tag{3}$$

where $c, a_0, \ldots, a_{n-1}$ are real constants and the knots $\{\xi_i\}$ obey the constraints $0 < \xi_1 < \xi_2 < \ldots < \xi_{r-1} < 1$.

Manifestly, the perfect spline $P(x)$ exhibits the property that its nth derivative, though changing sign at each knot ξ_i, maintains a constant absolute value, in this case $|c|n!$ The key to the solution of the problem (2) is the following interpolation theorem.

THEOREM 1. *Let $\{x_i\}_{i=1}^{n+r}$ be prescribed with $0 \leqq x_1 \leqq x_2 \leqq \cdots \leqq x_{n+r} \leqq 1$ involving no coincident block exceeding n points. Let $\{\alpha_i\}_{i=1}^{n+r}$ be given real data. Then there exists a perfect spline $P(x)$ of the form* (3) *with at most $k - 1$ knots in* $[0, 1]$, *$k \leqq r$, such that*

$$P(x_i) = \alpha_i, \qquad i = 1, 2, \ldots, n + r. \tag{4}$$

AMS 1970 *subject classifications*. Primary 46E35.