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COMPACTNESS IN LOCALLY COMPACT GROUPS[1]

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In [**4**], Glicksberg proved that every $\omega(G, \hat{G})$-compact subset of a LCA group $(G, \mathscr{T})$ is $\mathscr{T}$-compact. In this note three generalizations of this result are given (viz., Theorem 1, Theorem 3 and Theorem 4). In each case, a set is given two comparable topologies whose compacta are shown to coincide. It follows (with the weaker topology T_2) that sequential convergence also coincides in the two topologies.

THEOREM 1. *Let $(G, \mathscr{T})$ be a locally compact T_2 group (not necessarily abelian) with $\hat{G}$ the continuous irreducible unitary representations of G. Then every $\omega(G, \hat{G})$-compact subset of G is $\mathscr{T}$-compact. (For each π in $\hat{G}$, the set of unitary operators on H^π is given the weak operator topology.)*

The separable metric case of Theorem 1 is proved first. A countable separating subfamily of $\hat{G}$ is shown to exist and an argument of Eberlein used to prove that every weakly compact set is weakly sequentially compact. Ernest proved [**3**, Corollary 4.5] that every weakly convergent sequence is $\mathscr{T}$-convergent, so this case of the proof is complete, since $\mathscr{T}$ is metric. The σ-compact case is shown to follow by virtue of the fact that a compact normal subgroup can be factored out to leave a separable metric quotient group. The general locally compact T_2 case is then established by showing (via irreducible positive definite functions) that every weakly compact set must lie in an open σ-compact subgroup of G.

The following theorem settles the question raised and partially answered by Bichteler in [**1**]. It is an immediate consequence of Theorem 1.

THEOREM 2. *Let $\mathscr{T}_1$, $\mathscr{T}_2$ be locally compact Hausdorff topologies on a group G, which give rise to the same continuous irreducible unitary representations of G. Then $\mathscr{T}_1 = \mathscr{T}_2$.*

THEOREM 3. *Let $(G, \mathscr{T})$ be a locally compact T_2 group and let $P(G)$ be the set of continuous positive definite functions on G. Then every subset of $P(G)$ compact in the topology of pointwise convergence is compact in the compact-open topology.*

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[1] This research comprises the author's doctoral dissertation written at the University of North Carolina at Chapel Hill under the direction of Professor Richard Iltis and Professor B. J. Pettis.