BULLETIN OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 79, Number 1, January 1973

THE MORSE LEMMA ON ARBITRARY BANACH SPACES

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Communicated by Everett Pitcher, May 19, 1972

In [**4**], the author proved the Morse lemma on a real Banach space E which is the dual space of some space E_0, as for example the Sobolev spaces L_p^k, $k \geqq 0$, $1 < p < \infty$ and the Hölder spaces $C^{k,\alpha}$ [**5**]. The author's first result extended earlier versions by Morse and Palais [**2**], [**3**]. In this note we state a theorem and sketch a proof of the Morse lemma for *any* Banach space.

Let $f : U \to R$ be at least C^3 (3 times differentiable) with $0 \in U$ a critical point of f ($Df_0 = 0$). By the Taylor theorem we can write f as

$$f(x) = \tfrac{1}{2}\langle A_x x, x\rangle + f(0)$$

where $A : U \to L(E, E^*)$ {the linear maps from E to E^*} is C^1 and symmetric; i.e.,

$$\langle A_x u, v\rangle = \langle A_x v, u\rangle \qquad \forall u, v \in E.$$

Here $\langle A_x u, v\rangle$ denotes the standard bilinear pairing of E and E^*.

DEFINITION. 0 is said to be a nondegenerate critical point if

(1) $\exists$a nbhd $N \subset U$ of 0 and constants C_1 and C_2 so that $\forall t, t', t_1, t_2 \in N$.

(a) A_t^* is injective (thus A_t is injective).

(b) $\|DA_t(h)(y)\| \leqq C_1\|h\| \cdot \|A_{t'}y\|$ for all $h, y \in E$.

(c) $\|DA_{t_1}(h)(y) - DA_{t_2}(h)(y)\| \leqq C_2\|h\| \cdot \|t_1 - t_2\|\ \|A_{t'}y\|$ for all h, $y \in E$, where D denotes the Fréchet derivative of A with respect to the subscript variable.

(2)(a) For each $t \in N$, $\langle A_t x_n, y\rangle$ converges to zero for all y iff $\langle A_0 x_n, y\rangle$ converges to zero for all y.

(b) Given $t \in N$ if $\langle A_t x_n, y\rangle$ converges to zero for all $y \in E$ then $\langle DA_t(h)(x_n), y\rangle$ converges to zero of all $y \in E$ and $h \in E$.

It is not difficult to check that if $E = H$ (Hilbert space) and $A_0 : H \to H$ is an isomorphism (the standard definition of nondegeneracy) then conditions (1) and (2) are satisfied.

THEOREM (MORSE LEMMA). *Let $f : U \to R$ be C^3 with $0 \in U$ a nondegenerate critical point of f. Then there exists a local diffeomorphism ϕ of a nbdh of 0 so that*

$$f \circ \varphi(x) = \tfrac{1}{2}D^2 f_0(x, x) + f(0),$$

where $D^2 f_0$ is the second derivative of f at 0.

AMS (MOS) subject classifications (1970). Primary 49F15, 58E05.