BULLETIN OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 79, Number 1, January 1973

# GROUPS OF INVERTIBLE ELEMENTS OF BANACH ALGEBRAS[1]

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Communicated by Mary Ellen Rudin, July 20, 1972

ABSTRACT. Let $A$ be a complex Banach algebra, $G$ its group of invertible elements, and $G_e$ the component of the identity of $G$. Then $G_e$ is a closed, normal subgroup of $G$. This paper contains examples of $B^*$ algebras $A$ for which $G/G_e$ is finite, but not trivial, and of a $B^*$ algebra for which $G/G_e$ is noncommutative.

Let $A$ denote a complex Banach algebra, $G$ its group of invertible elements, and $G_e$ the component of the identity of $G$. If $A$ is finite-dimensional, or if $A = B(H)$, the algebra of all bounded linear operators on a Hilbert space $H$, or if $A$ is commutative, then $G/G_e$ is torsion free. For the first two cases we actually have $G$ connected, so $G = G_e$. A proof of the last result, which is due to Lorch, can be found in [**3**, p. 15]. We shall give examples of closed, noncommutative subalgebras of $B(H)$ for which $G/G_e$ is finite, but not trivial, and of a $B^*$ algebra for which $G/G_e$ is not abelian. Our examples will be special cases of the following class of Banach algebras.

Let $m$ be a finite, positive Borel measure whose support is a compact Hausdorff space $X$. Let $A(X, n)$ denote the set of all continuous functions from $X$ into $M_n$, the algebra of all complex $n \times n$ matrices. Then $A(X, n)$ is a Banach algebra under the pointwise addition and multiplication of functions and the following norm:

$$\|F\| = \sup_{x \in X} |F(x)|, \qquad F \in A(X, n),$$

where

$$|F(x)| = \sup\left\{|F(x)y| : y \in \boldsymbol{C}^n, \sum_{i=1}^{n} |y_i|^2 \leqq 1\right\}.$$

We can also define an involution on $A(X, n)$ by

$$F^*(x) = (F(x))^* \quad \text{for } F \in A(X, n) \text{ and } x \in X,$$

where $(F(x))^*$ denotes the conjugate transpose of $F(x)$. Then $A(X, n)$ is a $B^*$-algebra under this norm and involution. For, each $F \in A(X, n)$ induces an operator $\tilde{F}$ on $H = L^2(m) \oplus \cdots \oplus L^2(m)$ by

$$(\tilde{F}f)(x) = F(x)f(x) \quad \text{for } f \in H \text{ and } x \in X.$$

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*AMS (MOS) subject classifications* (1970). Primary 46L05, 46D10; Secondary 55E40.
*Key words and phrases*. Banach algebra, torsion, homotopy group.
[1] This research was supported by N.S.F. Grant GP-24182.
[2] The author would like to thank Professor E. Faddell for his help.