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RADICAL BEHAVIOR AND THE WEDDERBURN FAMILY

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1. **A question.** One sometimes constructs a family of algebras (e.g. various group algebras) and then hopes to prove that all its members have zero radical. If this is false, then one may attempt to describe the radicals that occur. Here we discuss the reverse process. Encouraged by the dictum 'When you have a lemon, make a lemonade,' we identify the following problem. *Given a nonzero nilpotent algebra N, describe the set of unital algebras A satisfying the equation*

$$\text{rad}\, A = N \tag{1.1}$$

(*together with a certain nontriviality condition*; see (2.1)). If the underlying scalar field k is perfect, then the Wedderburn Principal Theorem implies that (1.1) is equivalent to the search for semisimple S whose multiplication "associates" with that of N so that the semidirect sum

$$A = N + S \tag{1.2}$$

is associative (and has no "useless" semisimple ideal summands; see (2.1)). *Thus we are curious about nontrivial extensions of the trivial process of adjoining a unit to an algebra that lacks one.*

Some basic intuitions about (1.2): (a) If S is to be complicated, then the given N should be relatively uncomplicated. For instance, if S has orthogonal idempotents e_α, e_β, then the subspace $e_\alpha \cdot N \cdot e_\beta$ must have zero square (very uncomplicated). (Is there a conservation law?) (b) If a more complicated ("generic") nilpotent algebra associates with the semisimple S, then every less complicated nilpotent specialization should do likewise. (c) The collection of *maximal* S satisfying (1.2) is a reasonable structural invariant, yielding insight into the overall decomposability of N as an algebra with operators.

The main results announced here: Theorem (2.6) relating solutions S of (1.2) for a fixed N to solutions for its graded form $\text{gr}\, N$; Theorems (3.1) and (4.1), which solve (1.2) in the cases of commutative indecomposable N and square-zero N, respectively (see also (6.1) for maximal S); the deformation theorem (5.1) which makes precise our intuition (b) above; the stability result (7.2) relating idealhood in N with that in $A = N + S$, and reducing the general problem (1.2) to the case of indecomposable N;

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