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TRACE CLASS, WIDTHS AND THE FINITE APPROXIMATION PROPERTY IN BANACH SPACE

BY R. A. GOLDSTEIN AND R. SAEKS

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I. **Introduction.** Although there have been a number of attempts [3] to define operator classes in Banach space whose properties are analogous to the classical trace class operators of Hilbert space [1], [2] it is generally agreed that a satisfactory definition has yet to be achieved [3]. The purpose of the present note is to introduce a new approach to the problem wherein operator widths [2], [4] in Banach space replace the eigenvalues of the Hilbert space formulation; the viability of the approach being illustrated by the formulation of a number of sufficient conditions for an operator to have the finite approximation property in terms of its widths. Moreover, unlike the previous approaches [3] the trace class operators defined via operator widths are representation independent and coincide exactly with the classical definitions in Hilbert space.

II. **Definitions and results.** In the sequel X is a Banach space normed by $\|\cdot\|$, B is the unit ball in X and $\mathscr{L}_n$ is the set of n-dimensional subspaces of B. The nth width, $d_n(A)$, of an operator A on X is defined [4] by

$$d_n(A) \equiv \inf_{L \in \mathscr{L}_n} \sup_{u \in B} \inf_{v \in L} \|Au - v\|. \tag{1}$$

Classically, Kolmogorov [4] defined the nth width of a set to be a measure of the degree to which the set could be approximated by n-dimensional subspaces, the definition of equation (1) being that of Kolmogorov applied to $A(B)$.

Some remarks concerning the sequence $\{d_n(A)\}$ are as follows:

(a) $d_0(A) = \|A\|$;

(b) $\{d_n(A)\}$ is a nonincreasing sequence and $d_n(A) \to 0$ iff A is a compact operator;

(c) (see for example [2]) for X a Hilbert space and A a compact linear operator on X, set $s_n(A) \equiv \lambda_n((A^*A)^{1/2}) \equiv$ the nth eigenvalue of $(A^*A)^{1/2}$ ($n = 1, 2, \ldots$) (these are called the s numbers or characteristic numbers of A). Then $d_n(A) = s_{n+1}(A)$ ($n = 0, 1, 2, \ldots$). A is an Hilbert-Schmidt or nuclear operator if the sequence $\{s_n(A)\}$ is an l_2 or l_1 sequence.

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