BULLETIN OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 78, Number 5, September 1972

SINGULARITY SUBSCHEMES AND GENERIC PROJECTIONS

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Communicated by Michael Artin, April 3, 1972

Let k be an algebraically closed field, and let P^n be projective n-space over k. Let $V^r \subset P^n$ be a smooth projective variety. It is known that V can be embedded in P^{2r+1} (cf. [2]), but there are smooth r-dimensional varieties which cannot be embedded in P^{2r}.

Let $r \leqq m \leqq \min(2r, n-1)$, and let $\pi : V \to P^m$ be induced by projection from an $(n-m-1)$-subspace $L \subset P^m$ with $L \cap V = \emptyset$. As in [4], we ask what can be said about π when L is chosen *generically*, i.e. L is chosen from some dense open subset of the corresponding Grassmann variety. In the case that $m \geqq r+1$, the problem is to describe the local nature of the singular locus of $V' = \pi(V)$. This is of interest because V' can be chosen to be birational to V. Specifically, we would like to describe the structure of the local rings $\hat{\mathcal{O}}_{V',y}$ for closed points $y \in V'$.

For $i > 0$, let $S_i \subset V$ consist of all points x at which the tangent map has rank $\leqq r - i$. Thus

$$S_i = \{x \in V \mid \dim_{k(x)}(\Omega^1_{X/P_m}(x)) \geqq i\}.$$

The following result is known; cf. [5, Lemma 3].

PROPOSITION. *If L is chosen generically, then S_i is of pure codimension $i(m-r+i)$ in V, for all $i > 0$.*

In particular, if $m = r+1$, then $\operatorname{codim}(S_1) = 2$, and $\operatorname{codim}(S_2) = 6$. This says that $S_2 = \emptyset$ if $r \leqq 5$ and $m \geqq r+1$.

Let x be a closed point of $S_1 - S_2$, and let $y = \pi(x)$. Let $\pi^* : \mathcal{O}_{P^m,y} \to \mathcal{O}_{V,x}$ be the corresponding homomorphism of local rings. We can choose parameters $t_1, \ldots, t_r$ (resp. $u_1, \ldots, u_m$) in $\mathcal{O}_{V,x}$ (resp. $\mathcal{O}_{P^m,y}$) such that $\pi^*(u_i) = t_i$ for $i = 1, \ldots, r-1$, while $\pi^*(u_i) \in \mathfrak{m}_x^2$ for $i = r, \ldots, m$, where $\mathfrak{m}_x \subset \mathcal{O}_{V,x}$ is the maximal ideal. In a natural way, one can define closed subschemes $S_1^{(q)} \subset V - S_2$ such that if $\operatorname{char}(k) = 0$, the local generators of the sheaf of ideals defining $S_1^{(q)}$ are $(\partial^j u_i/\partial t_r^j)$ for $1 \leqq j \leqq q$, and $r \leqq i \leqq m$. In general, there are differential operators $D^{(j)} : \mathcal{O}_{V,x} \to \mathcal{O}_{V,x}$ such that $(D^{(j)}f)(x)$ is the coefficient of t_r^j in the power series expansion of f (cf. [1, §16]). The elements $D^{(j)}(\pi^* u_i)$ are the correct local generators.

AMS 1970 *subject classifications*. Primary 14B05, 14N05; Secondary 14M15.

Key words and phrases. Projective algebraic variety, generic projection, differential operator, Grassmann variety, Schubert cycle.

[1] Supported by NSF Grant GP-20550.