BULLETIN OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 78, Number 3, May 1972

WHICH ABELIAN GROUPS CAN BE FUNDAMENTAL GROUPS OF REGIONS IN EUCLIDEAN SPACES?[1]

BY BAI CHING CHANG

Communicated by Emery Thomas, November 10, 1971

Let C_n denote the collection of all abelian groups that can be fundamental groups of regions in S^n. It is clear that $C_k \subseteq C_{k+1}$. It is also easy to see that C_1 and C_2 each consist of just two groups—the trivial groups 1 and the infinite cyclic group Z. We shall see in this paper that actually $C_k = C_{k+1}$ for $k \geqq 4$, so we shall be concerned mainly with the difference between regions in S^3 and regions in S^4.

If a region A in S^n is not S^n itself, we may assume that $A \subset R^n$, and that there is a point e of A that is at a distance $\geqq 1$ from $R^n - A$. Using barycentric subdivision T_k of R^n of mesh converging to zero, where T_l is a refinement of T_k if $l < k$, let U_k be the interior of the union of those simplexes that lie in A and are at a distance $\leqq k$ from e. Take A_k to be the component of U_k that contains e_i. It is easy to see that $A_l \leqq A_k$ if $l < k$, and that $\bigcup_{k=1}^{\infty} A_k = A$; thus $\pi(A)$ is equal to the direct limit of the sequence $\{\pi(A_k)\}$. Since each $\pi(A_k)$ is finitely generated, $\pi(A)$ must be countable.

Now suppose that $G = \pi(A)$ is abelian. Since $G_i = \pi(A_i)$ is finitely generated, the image K_i of G_i in some $G_s = \pi(A_s)$ of the inclusion $G_i \to G_s$ must be abelian. Replacing the sequence $\{G_i\}$ by a subsequence if necessary, we may assume that the image K_i of G_i in G_{i+1} is abelian.

The calculation of C_3 is closely related to the following problem: "Which elements of a link group commute?" In fact, if we use brick subdivision instead of barycentric subdivision of R^3 in the construction of A_k, we may assume that each $S^3 - A_k$ is the union of a finite number of handle-bodies-with-knotted-holes, semilinearly imbedded in S^3. Since each G_k is finitely generated, so is its abelianized group $\bar{G}_k = H_1(A_k)$. We can find non-singular loops $\{x_1, \ldots, x_p\}$ that generate $H_1(A_k)$. By the Alexander duality theorem and the fact that $S^3 - A_k$ is a manifold, we can also find non-singular loops $\{y_1, \ldots, y_p\}$ in $S^3 - A_k$ which are dual to $\{x_1, \ldots, x_p\}$ in the sense that the linking number (x_i, y_i) between x_i and y_i is equal to δ_{ij}, where δ_{ij} is the Kronecker delta. The image of any two elements of G_{k-1} in G_k must commute in the complement of the link $y_1 \cup y_2 \cup \cdots \cup y_p$.

The following theorem (cf. [**6**] and [7]) makes it possible to deal with arbitrary links.

AMS 1969 *subject classifications*. Primary 5520, 5705.

[1] This paper represents a portion of the author's Ph.D. thesis, written under the direction of Professor Ralph H. Fox at Princeton University.