BULLETIN OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 77, Number 3, May 1971

CLASSIFICATION OF SYMBOL SPACES OF FINITE TYPE

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Communicated by Stephen Smale, November 18, 1970

Introduction. Symbol spaces and their shift maps have a long history in dynamics beginning at the latest in 1921 with Morse [6]. Recently the work of Smale [10], Bowen-Lanford [3] and especially Bowen [1], [2], and Alekseev [0] shows that *subshifts of finite type* are crucial in the study of diffeomorphisms satisfying Smale's Axiom A. Thus the problem of classifying these elementary dynamical systems is basic to the geometric study of diffeomorphisms.

In this setting, Bowen-Lanford show how to attach a square matrix A (of zeros and ones) to each shift of finite type. But this actually dates from the earlier papers [7], [8] of Parry on intrinsic Markov chains. The problem is to classify, up to conjugacy, the shift maps of a shift space of finite type. Obviously the number of periodic points of each period is an invariant. All of these numbers are contained in the zeta function, shown to be the reciprocal of $\det(I-tA)$ by Bowen-Lanford.

But the zeta function is not enough—see Example 3 below. Something like similarity ($A \sim PAP^{-1}$) over the integers (actually positive integers) is necessary. The fact that this is in general not possible—the inverse of a positive matrix, even if it exists and is integral, generally has negative entries—indicates the need for a more subtle equivalence relation.

We use shift equivalence (over Z^+). This is the same type of equivalence which was used in an earlier paper [11] to classify one-dimensional attractors. In fact, we feel that some sort of "shift equivalence" will be involved for any classification of *basic sets* of Axiom A diffeomorphisms. The recent work of Sinai [9] and Bowen [1], showing that each basic set is a quotient in a nice way of subshift of finite type, is a major step in their classification. I have just been informed that Alekseev has also proved this theorem.

AMS 1970 *subject classifications*. Primary 58F15, 58F20, 54H20.

Key words and phrases. Symbol space, shift of finite type, Bernoulli shift, intrinsic Markov chain, topological conjugacy, basic set, Axiom A, zeta function, shift equivalence, baker's map.

[1] We received partial support in this research from NSF Grant GP 9637. We would also like to thank C. Zeeman and L. Motchane for their hospitality at the University of Warwick, Mathematics Institute, and the Institut des Hautes Études Scientifiques, Bures-sur-Yvette, respectively.