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BIHARMONIC CLASSIFICATION OF RIEMANNIAN MANIFOLDS[1]

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We consider a noncompact orientable C^∞ manifold R of dimension $m \geqq 2$ with a C^∞ Riemannian metric $ds^2 = \sum_{i,j=1}^m g_{ij}\, dx^i\, dx^j$. The corresponding Laplace-Beltrami operator is

$$(1)\qquad \Delta\,\cdot\, = -g^{-1/2} \sum_{i=1}^m \frac{\partial}{\partial x^i}\left(\sum_{j=1}^m g^{1/2} g^{ij} \frac{\partial\,\cdot}{\partial x^j}\right),$$

with $g = \det(g_{ij})$ and $(g^{ij}) = (g_{ij})^{-1}$. Let φ and ψ be continuous functions on the Wiener or Royden boundary β of R (cf. [2] and [5]; the terminology used in the present note is adopted from [5]). We are interested in solving the biharmonic boundary value problem

$$(2)\qquad \Delta^2 u = 0, \qquad u \,|\, \beta = \varphi, \qquad \Delta u \,|\, \beta = \psi.$$

Here u and Δu are required to be continuously extendable to the Wiener or Royden compactification R^* of R. If $\varphi = \psi = 0$ on the harmonic boundary $\delta \subset \beta$, then the solution of (2) vanishes identically. The problem, therefore, is to solve

$$(3)\qquad \Delta^2 u = 0, \qquad u \,|\, \delta = \varphi, \qquad \Delta u \,|\, \delta = \psi.$$

Such problems are of fundamental importance in applications of biharmonic functions to physics, in particular the theory of bending of thin plates (cf. [1]). Unconditional solvability cannot be expected, since the solution of (3) clearly exists for the unit ball R in the Euclidean space E^m but not for $R = E^m$. This simple example illustrates the significance of the biharmonic classification problem in the theory of elasticity. Here we shall announce some results on the problem; the proofs will appear elsewhere (cf. [3], [4]).

1. Quasiharmonic functions. The simplest nonharmonic biharmonic functions are what we shall call *quasiharmonic functions* u, characterized by $\Delta u = \text{const} \neq 0$. The normalized class $Q = Q(R) = \{u \in C^2(R) \,|\, \Delta u = 1\}$ can be viewed as a special class of super-

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