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A NOTE ON NONRIGID NASH STRUCTURES

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1. Introduction. Geometry in the Nash category is intermediate between real algebraic and real analytic geometry, and provides a link whereby methods of algebraic geometry can be applied to some problems in differential topology. See [1] for example.

The Nash structure of a manifold embedded in $\boldsymbol{R}^n$ is not deformed by perturbations of the embedding; this is a consequence of the uniqueness theorem of Nash [2]. In the context of the general problem of deformation of Nash structures, Mazur has asked whether these "embedding" structures are in fact rigid (see below for definitions). We show here that for the unit circle S^1 this is not the case.

We wish to thank Barry Mazur for introducing us to Nash manifolds, and we are most grateful to him for the several conversations in which he showed us how to present our own examples.

2. Definitions. A real analytic function $f: U \to \boldsymbol{R}$ on an open set $U \subset \boldsymbol{R}^n$ is called *algebraic* if there is a nontrivial real polynomial P such that $P(x_1, \cdots, x_n, f) = 0$ where the x_i are the coordinate functions. A *Nash manifold* is a topological space M together with a sheaf $\mathcal{O}_M$ of local rings over M such that every point $p \in M$ has a neighbourhood V on which $\mathcal{O}_M | V$ is isomorphic to the sheaf $\mathcal{O}_U$ of germs of algebraic functions on open subsets of an open subset U of $\boldsymbol{R}^n$. Alternatively, a Nash manifold can usefully be thought of as a real analytic manifold with maximal atlas $\{U_\alpha, \phi_\alpha\}$ of charts such that the overlap maps $\phi_\beta \circ \phi_\alpha^{-1}: \phi_\alpha(U_\alpha \cap U_\beta) \to \phi_\beta(U_\alpha \cap U_\beta)$ are all coordinate-wise algebraic. A real analytic map $\psi: M \to N$ is a *Nash map* if $\psi^* \mathcal{O}_N \subset \mathcal{O}_M$. Nash manifolds M, N are *equivalent* $(M \cong N)$ if there is a diffeomorphism $h: M \to N$ such that h and h^{-1} are Nash maps. A Nash manifold M is *rigid* if any real analytically locally trivial Nash map $p: E \to B$ with $p^{-1}(b) \cong M$ for some $b \in B$ is locally trivial (in the Nash sense) at b.

EXAMPLES. (1) $\boldsymbol{R}^n$ has a canonical Nash structure $\boldsymbol{R}^n_c$; we take $U = \boldsymbol{R}^n$ in the above.

(2) A real algebraic embedding $i: M \to \boldsymbol{R}^n$ induces an *embedding structure* M_i on M from $\boldsymbol{R}^n_c$ by $\mathcal{O}_{M_i} = i^* \mathcal{O}_{R^n_c}$. This structure does not depend on the choice of embedding [2] so we denote it by M_{emb}.

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