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BOUNDEDNESS AND OSCILLATION OF SOLUTIONS OF THE LIÉNARD EQUATION

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1. **Introduction.** Consider the generalized Liénard differential equation

$$x'' + f(x)x' + g(x) = e(t) \tag{1}$$

of which

$$x'' + f(x)x' + g(x) = 0 \tag{2}$$

is a special case. We will assume throughout that f and g are continuous and e is sectionally continuous. Many papers have been written giving sufficient conditions for boundedness or oscillation of solutions of (1) or (2), and surveys of results known prior to 1963 can be found in Sansone and Conti [**3**] and Reissig, Sansone, and Conti [**2**].

The study of equations (1) and (2) has fallen into two cases: first, the case where $f(x) \geqq 0$ for all x, and secondly, the case where $f(x) < 0$ for $|x|$ small. The best results for the first case were given by Burton and Townsend [**1**] in the form of *necessary and sufficient* conditions for boundedness and oscillation of solutions of (1) in the case $xg(x) > 0$ for $x \neq 0$. The results to be described here cover both of the cases mentioned above and generalize all results, including those in [**1**], which are known to the present time. The restrictions on the sign of $f(x)$ will be removed and instead placed on its integral. Furthermore, the conditions on the divergence of the integrals of f and g will be relaxed from those asked by previous authors.

2. **Uniform ultimate boundedness.**

DEFINITION. The solutions of (1) are uniformly ultimately bounded if there exists a constant $D > 0$ such that for any solution $x(t)$ there is a time T such that $|x(t)| < D$ and $|x'(t)| < D$ for all $t > T$.

Define $F(x) = \int_0^x f(s)\, ds$ and $E(t, t_0) = \int_{t_0}^t e(s)\, ds$ and write (1) as the system

$$\begin{aligned} x' &= y - F(x) + E(t, t_0), \\ y' &= -g(x). \end{aligned} \tag{3}$$

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