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A RELATION BETWEEN TWO SIMPLICIAL ALGEBRAIC K-THEORIES

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There is a proliferation of proposed algebraic K-theories [5], [6], [8], [11], [12], [13], [15] and one of the present authors can share the blame for three of them. However some rather striking relationships have been found which indicate that the various K-theories, while not the same, are at any rate comparable. This note describes a relation between the K-theory proposed by Quillen [13], which has the advantage of computability using powerful techniques of the homology of groups, and that K-theory defined axiomatically in [8] and constructed semisimplicially in [5], which possesses extremely pleasant functorial properties. It is our hope that this connection will be useful in computing the K-theory of [8], and thus eventually the stable K-theory [7] which is analogous, in this rarefied setting of rings, with stable homotopy theory.

We begin by recalling (in slightly different form from [13]) Quillen's construction. For any ring R, one forms $\mathbf{Z}_\infty \overline{W}(\mathrm{Gl}(R))$. Here $\mathrm{Gl}(R)$ is regarded as a (constant) simplicial group, $\overline{W}$ is the simplicial classifying space, [10, p. 87], and $\mathbf{Z}_\infty$ is the integral completion functor of Bousfield and Kan [2]. Then $K_i^Q(R) = \pi_i(\mathbf{Z}_\infty \overline{W}\mathrm{Gl}(R))$, $i \geqq 1$, where the superscript refers to the author.

In order to give the simplicial definition of [5] of the K-theory of [8], we recall some terminology. One works in the category *ring* of rings (without unit) and one lets E be the endomorphism of *ring*, $ER = tR[t]$, the *path ring*. The morphisms $\epsilon: E \to I$, $\mu \to E^2$ given by

$$\epsilon_R: ER = tR[t] \to R, \qquad \text{"}t \to 1,\text{"} \quad \text{and}$$
$$\mu_R: ER = tR[t] \to tuR[t, u] = E^2R, \qquad t \to tu,$$

give rise to the cotriple (E, ϵ, μ) in *ring*. Let $\overline{E}R$ be the augmented semisimplicial complex, $(\overline{E}R)_n = E^{n+2}R$, $n \geqq -1$, constructed from this cotriple, and set

$$K^{-i}(R) = \tilde{\pi}_{i-2}(\mathrm{Gl}(\overline{E}R)), \qquad i \geqq 1.$$

The upper indexing is motivated by topological considerations, and

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