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DIRICHLET FINITE SOLUTIONS OF $\Delta u = Pu$, AND CLASSIFICATION OF RIEMANN SURFACES[1]

BY MITSURU NAKAI[2]

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1. Problem and its background. Consider a C^1 differential $P(z)\,dx\,dy$ $(z=x+iy)$ on an open Riemann surface R with $P(z) \geqq 0$. We denote by $PX(R)$ the set of C^2 solutions on R of the elliptic equation $\Delta u = Pu$, or more precisely, of $d^*du(z) = u(z)P(z)\,dx\,dy$, with a certain property X. For $P \equiv 0$ we use the traditional notation HX instead of OX. Let O_{PX} be the set of pairs (R, P) such that $PX(R)$ reduces to constants. Instead of $(R, P) \in O_{PX}$ we simply write $R \in O_{PX}$ if P is well understood. As for X we let B stand for boundedness, D for the finiteness of the Dirichlet integral $D_R(u) = \int_R du \wedge {}^*du$, and E for the finiteness of the energy integral $E_R(u) = D_R(u) + \int_R Pu^2\,dx\,dy$; we also consider combinations of these properties. It is known that

$$(1) \qquad O_G \underset{\neq}{\subset} O_{PB} \underset{\neq}{\subset} O_{PD} \subset O_{PBD} \subset O_{PE} = O_{PBE}.$$

Here O_G is the class of pairs (R, P) such that there exists no harmonic Green's function on R.

This type of classification problem was initiated by Ozawa [4] in 1952. It first came as a surprise when Myrberg [2] proved in 1954 the unrestricted existence of the Green's function for the equation $\Delta u = Pu$ $(P \not\equiv 0)$ for every R. This also eliminated the need of considering the nonexistence of nonnegative solutions in the case $P \not\equiv 0$. Following Myrberg's discovery, work in this direction largely pursued aspects which were different in nature from those in the harmonic case. Typically classes PD and O_{PD} were first considered by Royden [6] in 1959. Since the energy integral $E(u)$ for $\Delta u = Pu$ plays the same role as the Dirichlet integral $D(u)$ for the harmonic case, it is natural that PE and O_{PE} share properties of HD and O_{HD}. In this

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[2] The author's present address is the University of California, Los Angeles, California 90024.