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ABBREVIATING PROOFS BY ADDING NEW AXIOMS

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The purpose of this note is to state precisely and prove the following informal statement: If T is a theory and α is a new axiom such that $T+\text{non}\,\alpha$ is an undecidable theory then some theorems of T have much shorter proofs in $T+\alpha$ than in T. Notice that if T is an essentially undecidable theory, like e.g. arithmetic, this conclusion will be true provided α is a sentence which is not a theorem of T, since then $T+\text{non}\,\alpha$ is undecidable.

Let T be a formalized theory which among its logical functors has the negation $\neg$, the implication $\rightarrow$, and the alternative $\vee$. Let σ and τ be variables ranging over sentences formulated in the language of T and α one fixed such sentence. We denote by $\ulcorner\sigma\urcorner$ the Gödel number of σ, although here $\ulcorner\ \urcorner$ is just any one-to-one map of the set of sentences into the set of positive integers. For any theorem τ of T let $W(\tau)$ be also a positive integer measuring in some way the length of the shortest proof of τ in T. But all we need about $\ulcorner\ \urcorner$ and W are the following conditions:

(i) The set $\{2^n(2\ulcorner\tau\urcorner+1):\tau \text{ is valid in } T \text{ and } W(\tau)\leqq n\}$ is recursive.

(ii) There are recursive functions g and h such that for every σ

$$W(\alpha\rightarrow(\alpha\vee\sigma))\leqq g(\ulcorner\sigma\urcorner),\qquad h(\ulcorner\sigma\urcorner)=\ulcorner\alpha\vee\sigma\urcorner.$$

The meaning of (i) is that there is an algorithm to check if τ has a proof of length $\leqq n$. This stipulation entails that the set of Gödel numbers of the theorems of T is recursively enumerable. It is clear that reasonable $\ulcorner\ \urcorner$ and W satisfy (i) and (ii).

LEMMA. *If the theory $T+\neg\alpha$ is undecidable, i.e. the set $\{\ulcorner\sigma\urcorner:\alpha\vee\sigma$ is valid in $T\}$ is not recursive, then there is no recursive function f such that*

$$W(\tau)\leqq f(W(\alpha\rightarrow\tau)) \tag{1}$$

for every τ valid in T.

PROOF. Suppose to the contrary that (1) holds. We can assume without loss of generality that f is nondecreasing. Then by (1) and (ii) we get

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