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SOME FIXED POINT THEOREMS

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Introduction. We wish to summarize here some new asymptotic fixed point theorems. By an asymptotic fixed point theorem we mean roughly a theorem in functional analysis in which the existence of fixed points of a map f is proved with the aid of assumptions on the iterates f^n of f. Such theorems have proved of use in the theory of ordinary and functional differential equations (see [7], [8], [9] and [15]). It also seems likely that many of the fixed point theorems which have been used in nonlinear functional analysis can be unified by obtaining them as corollaries of general asymptotic fixed point theorems. These theorems also give new results, of course.

In our first section we restrict attention to continuous maps f defined on closed, convex subsets of Banach spaces. We obtain a general fixed point theorem (Theorem 1 below), and we prove that certain fixed point theorems of R. L. Frum-Ketkov [5], F. E. Browder [1], [2], W. A. Horn [6], G. Darbo [3], the author [11], [12], [13] and others follow as corollaries. In the second section we consider maps defined on more general spaces than closed, convex subsets of Banach spaces, and we generalize some of the results of §1.

1. We begin by recalling some notation from [11]. If U is a closed subset of a Banach space X, we shall say that $U \in \mathfrak{F}$ if there exists a closed, locally finite covering $\{C_j\}_{j\in J}$ of U by closed, convex sets $C_j \subset U \subset X$. We shall say that $U \in \mathfrak{F}_0$ if there exists a finite number of closed, convex sets $C_1, C_2, \cdots, C_n$ in X such that $U = \bigcup_{j=1}^n C_j$.

The basic lemma in all our work is the following geometrical result, which can be viewed as a generalization of a theorem of Dugundji [4]. If X below is a locally convex topological vector space, the same conclusions hold, with the exception that R may not be a retraction.

LEMMA 1. *Let C and D be closed subsets of a Banach space X with $C \supset D$. Assume that $C = \bigcup_{j=1}^n C_j$ and $D = \bigcup_{j=1}^n D_j$, where $C_j \supset D_j$ and C_j and D_j are closed, convex subsets of X for $1 \leqq j \leqq n$. Suppose that, for each subset $J \subset \{1, 2, \cdots, n\}$, $\bigcap_{j\in J} C_j$ is nonempty if and only if $\bigcap_{j\in J} D_j$*

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