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A NONCOMMUTATIVE EXTENSION OF THE PERRON-FROBENIUS THEOREM

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ABSTRACT. The existence and uniqueness of physical ground states is proved for various interactions in quantum field theory using some infinite dimensional extensions of the Perron-Frobenius theorem.

Let $\mathcal{A}$ be a von Neumann algebra with a finite regular trace. A bounded linear operator A from $L^2(\mathcal{A})$ to $L^2(\mathcal{A})$ is of *nonnegative type* if A takes nonnegative operators to nonnegative operators. The classical theorem of Perron and Frobenius [2] asserts that if $\mathcal{A}$ is finite dimensional (in which case $L^2(\mathcal{A}) = \mathcal{A}$) and commutative and if A is of nonnegative type and has spectral radius r then r is an eigenvalue of A and has a corresponding eigenvector which is nonnegative. If, in addition, A leaves no proper ideals of $\mathcal{A}$ invariant then r is an algebraically simple eigenvalue.

In this note we remove the assumptions that $\mathcal{A}$ is finite dimensional and commutative. Denoting by L_α and R_α the bounded operators of left and right multiplication on $L^2(\mathcal{A})$ by an element α in the von Neumann algebra $\mathcal{A}$ we put, for any projection e in $\mathcal{A}$, $P_e = L_e R_e$. P_e is a projection on $L^2(\mathcal{A})$. Its range will be called a Pierce subspace. We consider a bounded Hermitian operator A of nonnegative type on $L^2(\mathcal{A})$. We shall show that if, for some $p > 2$, A is bounded from $L^2(\mathcal{A})$ to $L^p(\mathcal{A})$ then $r = \|A\|$ is an eigenvalue of A and A has a nonnegative eigenvector with eigenvalue r. If, in addition, A leaves invariant no proper Pierce subspace then r has multiplicity one.

This work is motivated by our attempts to prove the existence and uniqueness of physical ground states in various models in quantum field theory. Among new results following from these methods is existence of the physical one particle rest state for some interactions involving Bosons, where $\mathcal{A}$ is commutative, and uniqueness of the physical vacuum for interactions involving Fermions, where $\mathcal{A}$ is a

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