KERNEL FUNCTIONS AND PARABOLIC LIMITS FOR THE HEAT EQUATION

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Let $D \subset \{(x, t): t > 0\}$ be a domain of the plane bounded by curves $x = \eta_1(t)$, $x = \eta_2(t)$, and $t = 0$, where $\eta_1(t) < \eta_2(t)$ for all t and, for each $T \in (0, \infty)$, $\eta_i(t)$ satisfies a Lipschitz condition with exponent $\frac{1}{2}$ on the interval $[0, T]$, $i = 1, 2$. Let $(X, T) \in D$ and $(y_0, s_0) \in \partial D$ with $s_0 < T$. A kernel function for the heat equation in D at (y_0, s_0) with respect to (X, T) is a nonnegative solution of the heat equation in D, $K(x, t)$, which vanishes continuously on $\partial D - \{(y_0, s_0)\}$ and is normalized by the requirement that $K(X, T) = 1$.

The notion of a kernel function has been studied in the case of harmonic functions in Lipschitz domains in $\boldsymbol{R}^n$ by Hunt and Wheeden [3], whose results include a representation theorem for nonnegative harmonic functions and a proof of the almost everywhere (with respect to harmonic measure) existence of finite nontangential boundary values for harmonic functions having a one-sided bound in a Lipschitz domain. (See also [2].) The present note describes analogous results for the heat equation in regions of the plane.

THEOREM. *If $(X, T) \in D$ and $(y, s) \in \partial D$ with $s < T$, then there exists a unique kernel function for the heat equation in D at (y, s) with respect to (X, T).*

It is clear that, for $s < T$, a kernel function at (y, s) with respect to (X, T) is completely determined by its values in $D_T = \{(x, t) \in D : t < T\}$. Thus, it suffices to consider kernel functions at (y, s) in the bounded region D_T. One is led to the following representation result.

THEOREM. *Let $\partial_p D_T$ denote the parabolic boundary of D_T, which is $\partial D_T \cap \{(x, t): t < T\}$, and for $(y, s) \in \partial_p D_T$, let $K(x, t, y, s)$ denote the value at (x, t) of the kernel function at (y, s) with respect to (X, T). If $u(x, t)$ is any nonnegative temperature in D_T, then there exists a unique regular Borel measure μ on $\partial_p D_T$ such that*

$$u(x, t) = \int_{\partial_p D_T} K(x, t, y, s)\, d\mu(y, s).$$

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