

# THE VITALI-HAHN-SAKS AND NIKODYM THEOREMS FOR ADDITIVE SET FUNCTIONS

BY R. B. DARST

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ABSTRACT. The purpose of this note is twofold. Firstly, we point out an appropriate version of the Vitali-Hahn-Saks and Nikodym theorems for finitely additive set functions defined on a sigma algebra of sets. Secondly, we apply our Theorem to extend a recent result of James K. Brooks for countably additive vector valued set functions to the general finitely additive case.

THEOREM. *Let  $\{\mu_n\}$  be a sequence of bounded and finitely additive scalar valued set functions defined on a sigma algebra  $\Sigma$  of subsets of a set  $S$ . If  $\mu(E) = \lim_n \mu_n(E)$  exists for every  $E \in \Sigma$ , then  $\mu$  is bounded and additive and the additivity of the  $\mu_n$  is uniform in  $n$ .*

In addition, suppose  $\lim_{\nu(E) \rightarrow 0} \mu_n(E) = 0$  for each  $n$ , where  $\nu$  is a nonnegative finitely additive set function defined on  $\Sigma$ . Then  $\lim_{\nu(E) \rightarrow 0} \mu_n(E) = 0$  uniformly in  $n$ .

Our Theorem follows from the weak convergence theory for finitely additive set functions. While a proof of it can be synthesized (modulo an observation) from [3], [5], [6] and [7], for the readers' convenience we shall also refer to the discussion of the work of Solomon Leader and Pasquale Porcelli [6] and [7] given in [4]. The corollary on p. 475 of [3] tells us that the sequence  $\{\mu_n\}$  is weakly convergent. Also, the  $(L)$ -space of bounded and additive functions on  $\Sigma$  is weakly complete [5, Theorem 12], so the sequence  $\{\mu_n\}$  converges weakly to  $\mu$ . Hence  $\mu$  is bounded and additive, and (cf. [4]) the weakly convergent sequence  $\{\mu_n\}$  is equi-absolutely continuous with respect to the bounded and additive function  $\varphi$  defined on  $\Sigma$  by

$$(1) \quad \varphi(E) = \sum_{k=1}^{\infty} 2^{-k} (1 + |\mu_k|(S))^{-1} |\mu_k|(E),$$

where  $|\mu_k|$  is the variation of  $\mu_k$ . Since the sequence  $\{\mu_n - \mu\}$  converges weakly to zero, Lemma 1 of [4] implies

$$(2) \quad \lim_j \left\{ \sup_k \left[ \sum_{i \geq j} |\mu_k(E_i) - \mu(E_i)| \right] \right\} = 0$$

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