FOLIATIONS OF CODIMENSION ONE

BY JOHN W. WOOD

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In this note we apply results of [6] to obtain some sufficient conditions for a plane field of codimension one on a manifold to be homotopic to a foliation. This and related questions on foliations are discussed in E. Thomas' survey [5, §4] and, for open manifolds, by A. Haefliger [2]. A. V. Phillips has shown [3] that any field of codimension one on an open manifold is homotopic to a foliation.

Let M be a compact riemannian manifold with boundary. It is convenient to work with the normal line field which corresponds to any plane field of codimension one. A line field is defined by a bundle monomorphism $f:\lambda \hookrightarrow \tau$ where λ is some line bundle over M and τ is the tangent bundle; we say λ embeds in τ. A homotopy of plane fields corresponds to a homotopy of bundle monomorphisms. We *require* $f(\lambda | \partial M)$ to be normal to the boundary and homotopies to be relative to the boundary. In particular, $\lambda | \partial M$ is trivial.

It is unknown which line bundles over M embed as the normal fields of foliations. We can however prove a stable theorem. Let $p:M\times S^1 \to M$ be the projection map.

THEOREM 1. *For any line bundle $\lambda \to M$, $p^*\lambda$ embeds as the normal field of a folidation of $M\times S^1$.*

This is in contrast to the situation in higher codimension. The normal bundle σ of a foliation must satisfy Bott's condition that the ring generated by the rational Pontrjagin classes of σ vanishes in dimension $>2 \dim\sigma$; and if σ does not satisfy Bott's condition neither does $p^*\sigma$. For codimension 2 for example $p_1(\sigma)^2 = 0$. If λ is the canonical line bundle over RP^m, then $p^*\gamma$ embeds as the normal field of a foliation of $RP^m \times S^1$ and $w_1(p^*\gamma)^m \neq 0$. This foliation is easily described. There is a map from the solid torus $B^m\times S^1$ onto $RP^m \times S^1$ which is a diffeomorphism on int $B^m \times S^1$ and a double cover from $S^{m-1}\times S^1$ to $RP^{m-1}\times S^1 \subset RP^m \times S^1$. The Reeb foliation of $B^m \times S^1$ passes to the desired foliation of $RP^m \times S^1$.

We will need the following known fact.

LEMMA 1. *Let $\lambda \to M$ be a line bundle, s a section transverse to the zero section, $N = s^{-1}$ (zero section), and $i: N \subset M$. Then* $w_1(\lambda) \cap [M] =$

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