THE EXISTENCE OF FREE METACYCLIC ACTIONS ON HOMOTOPY SPHERES

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Communicated by E. H. Brown Jr., April 27, 1970

This question is rather old [3]: Which groups having periodic cohomology can act freely on some homotopy sphere?

The first nontrivial restriction was supplied by Milnor in [3]. Every element of order two must lie in the center. Swan [6] showed that any group with periodic cohomology acts freely on a c.w. complex of the homotopy type of sphere.

If the group has odd order then it is metacyclic. The main result of this note is that most metacyclic groups of odd order *can* act freely and smoothly on some homotopy sphere (Corollary 6). Of independent interest is the discussion of the algebraic tools used in the solution.

I. **The algebraic tools.** Let π be a finite group and C_k be the category of pairs (M, φ) where

(1) M is a left $Z[\pi]$ module of homological dimension $\leqq 1$ and M as an abelian group has finite order and no two torsion.

(2) $\varphi: M \to \operatorname{Hom}_Z(M, Q/Z)$ is a $(-1)^k$ Hermitian form over $Z[\pi]$, i.e. φ is an isomorphism of $Z[\pi]$ modules and $\varphi(x, y) = (-1)^k \varphi(y, x)$, $x, y \in M$. $\operatorname{Hom}_Z(M, Q/Z)$ is made into an abelian group by $(\lambda f)(x) = f(\bar{\lambda} x)$ for $f \in \operatorname{Hom}_Z(M, Q/Z)$, $x \in M$, $\lambda \in Z[\pi]$ and $\bar{\lambda}$ is the conjugate of λ under the involution on $Z[\pi]$ defined by sending group elements into their inverses.

(3) $0 \to (M_1, \varphi_1) \to (M_2, \varphi_2) \to (M_3, \varphi_3) \to 0$ is exact in C_k iff $M = M_1 \oplus M_2$, $\varphi = \varphi_1 \oplus \varphi_2$.

Certain elements of C_k are to be regarded as trivial. These are obtained in this fashion: Let F be a free $Z[\pi]$ module of rank n and A an $n \times n$ matrix over $Z[\pi]$ satisfying $\bar{A}_{ij} = (-1)^k A_{ji}$. Define M by the exact sequence $0 \to F \xrightarrow{A} F \xrightarrow{\omega} M \to 0$ so that M is the cokernel of A. Suppose M is finite. Define a $(-1)^k$ Hermitian form φ on M by

$$\phi(\omega(x), \omega(y)) = \sum_{i,j} x_i \cdot \overline{A_{ij}^{-t}} \cdot \bar{y}_j \mod Z[\pi].$$

Explanation! Since M is finite, A has an inverse A^{-1} over $Q[\pi]$. Its transpose is A^{-t}. If $e_1 \cdots e_n$ is a base for F then $x = \sum x_i e_i$ and

AMS 1970 *subject classifications.* Primary 57D65, 57E25; Secondary 20C10.

Key words and phrases. Surgery, group actions on spheres, metacyclic groups, Hermitian forms, Brieskorn varieties.