ON CONTINUITY AND SMOOTHNESS OF GROUP ACTIONS

BY P. CHERNOFF AND J. MARSDEN

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In this note we give a short proof of a theorem of Bochner and Montgomery [1] using semigroup theory. In addition, we obtain more general results and give some applications (to diffeomorphism groups and nonlinear semigroups). A more detailed exposition will appear in [3].

1. Separate and joint continuity of group actions. The following generalizes a result of Ellis [6].

THEOREM 1. *Let M be a metric space, and let G be a Baire space with a group structure in which multiplication is separately continuous. Let $\pi: G\times M\rightarrow M$ be an action which is separately continuous. Then π is jointly continuous.*

PROOF. For each $x\in M$ there is a dense $\mathcal{G}_\delta$ $A\subset G$ such that π is continuous at (g_0, x) for $g_0\in A$. (See [4, p. 256, Problem 11] or [2, p. 255 Exercise 23].) For any $(g, x)\in G\times M$, we have, writing $\pi(g', x')=g'x'$,

$$g'x' = gg_0^{-1}g_0g^{-1}g'x' = \phi(g_0g^{-1}g'x')$$

where $\phi: M\rightarrow M$, $\phi(y)=gg_0^{-1}y$ and is continuous. But $g_0g^{-1}g'\rightarrow g_0$ as $g'\rightarrow g$, so as $g'\rightarrow g$, $x'\rightarrow x$ we get

$$g'x' = \phi(g_0g^{-1}g'x') \rightarrow \phi(g_0x) = gx$$

by joint continuity of π at (g_0, x).

If M is not metric the conclusion of Theorem 1 is no longer valid (consider G the circle and M the continuous functions on G with the topology of pointwise convergence).

As a corollary, it is not hard to deduce that a Baire separable metric group G for which multiplication is continuous is a topological group (compare [9] and [2, p. 258]). For example the C^k or H^s or $C^{k+\alpha}$ diffeomorphism group of a compact manifold satisfies these conditions; c.f. [5].

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