# ON THE FREDHOLM ALTERNATIVE FOR NONLINEAR OPERATORS

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Communicated March 4, 1970

Let $X$ be a locally convex topological vector space, $Y$ a real Banach space, $f$ a mapping (in general, nonlinear) of $X$ into $Y$. In several recent papers ([5], [6], [7]), Pohožaev has studied the concept of *normal solvability* or the *Fredholm alternative* for mappings $f$ of class $C^1$. If $A_x = f'_x$ is the continuous linear mapping of $X$ into $Y$ which is the derivative of $f$ at the point $x$ of $X$, $A_x^*$ the adjoint mapping of $Y^*$ into $X^*$, his principal results assert that if the nullspace $N(A_x^*)$ is trivial for every $x$ in $X$, and if one of the two following hypotheses hold:

(1) $Y$ is reflexive and $f(X)$ is weakly closed in $Y$;

(2) $Y$ is uniformly convex and $f(X)$ is closed in $Y$;

then the image $f(X)$ of $f$ must be all of $Y$.

It is our purpose in the present paper to considerably sharpen and generalize these results by use of a different and more transparent argument. In particular, we establish a corresponding theorem for an arbitrary Banach space $Y$ and $f(X)$ closed in $Y$, allow exceptional points $x$ in $X$ at which the hypothesis on $N(A_x^*)$ may not hold, and derive this theorem from a basic theorem on general rather than differentiable mappings. The techniques which we apply below may be extended to infinite-dimensional manifolds and may be localized to prove the openness of $f$ under stronger hypotheses (as we shall do in another more detailed paper).

To state our basic theorem, we use the following definition:

DEFINITION 1. *Let $X$ be a real vector space, $f$ a mapping of $X$ into the real Banach space $Y$, $x$ a point of $X$. Then the element $v$ of the unit sphere $S_1(Y)$ of $Y$ is said to lie in the set $R_x(f)$ of asymptotic directions for $f$ at $x$ if there exists $\xi \neq 0$ in $X$ and a sequence $\{\gamma_j\}$ of positive numbers with $\gamma_j \to 0$ as $j \to \infty$ such that for each $j$, $f(x+\gamma_j\xi) \neq f(x)$, while*

$$\|f(x+\gamma_j\xi) - f(x)\|^{-1}(f(x+\gamma_j\xi) - f(x)) \to v \qquad (j \to \infty).$$

Our basic general result is the following:

THEOREM 1. *Let $X$ be a real vector space, $Y$ a real Banach space, $f$ a*

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*AMS subject classifications*. Primary 4780, 4610; Secondary 4785.

*Key words and phrases*. Fredholm alternative, nonlinear operators, normal solvability, set of asymptotic directions, differentiable mappings, cone with interior, closed range, Bishop-Phelps theorem.