BOOK REVIEWS

Ergodic Properties of Algebraic Fields, by Yu. V. Linnik. Translated by Michael S. Keane. Band 45 of Ergebnisse der Mathematik und ihrer Grenzgebiete, Berlin, Springer-Verlag, 1968. $12.10

As one can see from the title, this book covers rather unusual material. Linnik has tried in it to connect two rather different branches of mathematics: Algebraic Number Theory and Ergodic Theory. It is a *tour de force* which he carries off quite well.

Previous works connecting ergodic theory with number theory have generally concentrated on proving results in metric number theory by using the ergodic theorem. One finds a suitable ergodic transformation (usually on the interval $[0, 1]$), constructs an invariant measure on it, looks at a reasonable function, and discovers via the ergodic theorem that almost every number has some property. For instance, one might let $T(x)=\{10x\}$ (where $\{a\}$ = fractional part of a). Then T is ergodic on $[0, 1]$, and Lebesgue measure is invariant under T. Let f be the characteristic function of $[.1, .2)$, and plug f and T into the ergodic theorem; out comes the statement that for almost every real number, a tenth of the digits in its decimal expansion are 1's.

Linnik is concerned with a different and subtler connection. Perhaps the best way to explain it is by means of an example (which, indeed, is the first example treated in the book). Let m be an integer $\equiv 1$ or $2 \pmod 4$ or $\equiv 3 \pmod 8$ and, consider the Diophantine equation $x^2+y^2+z^2=m$. This has primitive integral solutions (i.e., solutions where x, y, z have no common factor). What Linnik is after is an ergodic theorem concerning these primitive solutions. There are two immediate difficulties which arise. First of all, the ergodic theorem involves an ergodic transformation, and thus far there is no transformation at all on the solutions. A much more serious problem is that the number of primitive solutions is finite. It is not at all clear what can be meant by an ergodic transformation on a finite set.

The first problem is settled by constructing the transformation; the second needs more discussion. Linnik's eventual goal is to show that the primitive solutions are "uniformly distributed" on the sphere. It is not too hard to make sense of this notion. The primitive solutions are points on the 2-sphere $S_m=x^2+y^2+z^2=m$. Suppose that there are r_m such solutions. Let Λ be a sector from the origin cutting out the solid angle ω, and suppose that there are λ_m primitive solutions in the set $S_m \cap \Lambda$. Then the primitive points will be uniformly distributed if