ON MORSE THEORY AND STATIONARY STATES FOR NONLINEAR WAVE EQUATIONS

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The nonlinear equations to be discussed here can be written in the form

$$(1) \qquad \partial^2 u/\partial t^2 = Lu + N(x, u),$$

$$(1') \qquad i\, \partial u/\partial t = Lu + N(x, u)$$

where L is a second order elliptic formally selfadjoint differential operator acting on complex-valued functions $u(t, x)$ defined on $R^1 \times R^3$, and $N(x, u) = f(x, |u|^2)u$ is a complex-valued function jointly continuous in x and u with $f(x, r) = o(1)$ as $|r| \to \infty$. A complex-valued function $u(t, x)$ is called a stationary state of (1) [or (1′)] if

(a) $u(t, x)$ satisfies (1) [or (1′)] on $R^1 \times R^3$, and

(b) $u(t, x) = v(x)e^{i\lambda t}$ where λ is some real number and $v(x)$ is a smooth real-valued function defined on R^3, tending to 0 exponentially as $|x| \to \infty$ but not identically zero.

In this article we wish to examine the structure and properties of the stationary states of (1) [or (1′)] by combining recent results of Morse theory on Hilbert manifolds with concrete estimates for elliptic differential operators defined on R^3.

1. Statement of basic results. We begin with two affirmative facts concerning the existence of stationary states.

THEOREM 1. *Let* $L = \Delta - p^2$ (p = const.), $f(x, u) = k(|x|)|u|^\sigma$ *with* $0 < \sigma < 4$ *where* $k(|x|)$ *is a bounded positive continuous function uniformly bounded above zero. Then* (1) *and* (1′) *have (for each* $\lambda^2 < p^2$ *in* (1) *and* $\lambda < p^2$ *in* (1′)) *a countably infinite number of stationary states* $v_N(x, \lambda)$, $N = 0, 1, 2, \cdots$. *Each* $v_N(x, \lambda)$ *has precisely* N *nodal domains in* R^3 *and is nonoscillatory outside some fixed sphere of radius* R *(independent of* N*).*

THEOREM 2. *The* $v_N(x, \lambda)$ *of Theorem* 1 *(apart from a constant multiplier) are limits (as* $m \to \infty$*) of spherically symmetric nondegenerate critical points* $v_{Nm}(x, \lambda)$ *of index* N *of the functional* $\int_{B_m} k(|x|)|u|^\sigma u^2$ *on an infinite dimensional Hilbert manifold* $\mathfrak{M}_m$ *where* B_m *is a ball*

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