LIE ALGEBRAS OF ANALYTIC VECTOR FIELDS AND UNIQUENESS IN THE CAUCHY PROBLEM FOR FIRST ORDER PARTIAL DIFFERENTIAL EQUATIONS[1]

BY E. C. ZACHMANOGLOU

Communicated by C. B. Morrey, Jr., December 11, 1969

Let $P(x, D)$ be a partial differential operator defined in an open set $\Omega \subset R^n$ and let $x^0 \in \Omega$ be a boundary point of a closed subset F of Ω. We say that there is uniqueness in the Cauchy problem (UCP) for the system (P, x^0, F) if to every open neighborhood $U \subset \Omega$ of x^0 there is an open neighborhood $V \subset U$ of x^0 such that for every distribution u in U,

$$P(x, D)u = 0 \quad \text{in } U, \operatorname{supp} u \subset F \cap U \quad \Rightarrow u = 0 \quad \text{in } V.$$

The classical uniqueness theorem of Holmgren (as extended to distribution solutions by Hörmander [1]) gives a sufficient condition for UCP for the system (P, x^0, F) in the case in which P is a linear partial differential operator with analytic coefficients and the boundary of F is a C^1 hypersurface S. This condition is that S is not characteristic with respect to P at x^0. Although this condition is sufficient for UCP it is certainly not necessary. Malgrange [2], Hörmander [1], Trèves [3] and Zachmanoglou [4], [5], [6] have obtained some necessary and some sufficient conditions for UCP but the general problem is still unsolved.

In this note we present a necessary and sufficient condition for UCP for first order linear partial differential operators with analytic complex valued coefficients. No additional assumptions on the closed set F are made.

Let $\mathfrak{A}$ denote the ring of all real-valued analytic functions in Ω and let

$$(1) \quad P(x, D) = A + iB + c(x) = \sum_{j=1}^{n} a^j(x) D_j + i \sum_{j=1}^{n} b^j(x) D_j + c(x),$$

where $a^1, \cdots, a^n, b^1, \cdots, b^n$, Re c and Im c belong to $\mathfrak{A}$, $i = \sqrt{-1}$ and $D_j = \partial/\partial x_j$. A and B can be thought of as vector fields with coefficients in $\mathfrak{A}$. A trajectory of a collection $\mathfrak{C}$ of analytic vector fields is

AMS Subject Classifications. Primary 3501, 3530, 3537, 5736.

Key Words and Phrases. Partial differential equations, first order, uniqueness in the Cauchy problem, propagation of zeroes, Lie algebras, analytic vector fields, maximal integral manifold.

[1] This work was sponsored by the National Science Foundation Grant GP 12026.