B-SETS AND COLORING PROBLEMS

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The set-theoretic concept of "B-set" was first used by Bernstein in 1908 in dealing with a topological question. Since then it has appeared in number theory, combinatorics, and logic. We consider it in the realm of linear graphs, and in particular, questions of map coloring.

1. Definitions. Let $F=\{X_\alpha:\alpha\in A\}$ be a family of sets. A set B is called a B-set for this family if $B\cap X_\alpha\neq\emptyset$, all $\alpha\in A$, and $B\not\supseteq X_\alpha$, all $\alpha\in A$. A family F need not have a B-set. Observe that if B is a B-set for F, then so is the complement of B.

Consider a map covering S_2, the two-dimensional sphere. We shall assume that it is regular, that is, each vertex is of degree three. Each region of the map is a topological cell. Two regions are *adjacent* if they share at least one edge. A sequence of distinct regions R_1, $R_2, \cdots, R_n$, $n\geqq 3$, is a *cycle of length* n if R_i is adjacent to R_{i+1}, $1\leqq i\leqq n-1$, and R_n is adjacent to R_1. The cycle is *odd* or *even* according as n is odd or even.

2. Statement of results. It is well known that a map can be colored with four colors such that adjacent regions have different colors if and only if the regions can be partitioned into two sets, neither of which contains an odd-cycle. For purposes of comparison, we record this as

THEOREM 1. *A map can be colored with four colors if and only if the family of odd cycles has a B-set.*

This theorem suggests that we determine which families of cycles of regions in a map have a B-set. The next two theorems treat extreme cases.

THEOREM 2. *The family of cycles around vertices (hence of length 3) has a B-set if and only if the edges can be labelled r and b in such a way that at each vertex are one r and two b's.*

That such r, b labellings exist is a classical result of Petersen.

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