ON PARALLELISM IN RIEMANNIAN MANIFOLDS

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The definition of parallelism along a curve in a Riemannian manifold extends to higher dimensional submanifolds. This note is to announce a local existence and uniqueness theorem, Theorem B(p), for the extended definition. A proof of the theorem in the C^∞ category will appear in [2]. A proof, in the C^ω category, under somewhat weaker conditions, will appear in [1]. A global C^∞ version under stronger assumptions appears in [3]. This note ends with a sketch of a new proof of Theorem B(p).

Let $g: N^p \to M^m$ be a (not necessarily isometric) smooth (that is, C^∞ or C^ω) immersion of Riemannian manifolds. Let E be a euclidean vector bundle over N and F a euclidean vector bundle over M. A vector bundle map $G: E \to F$ is a *vector bundle isometry along* g provided that G sends the fibers $E(n)$ isometrically into the fibers $F(g(n))$. When E and F are the tangent bundles ($T(N^p)$ and $T(M^m)$), G is called a *tangent bundle isometry* (T.B.I.) *along* g. The *normal bundle to a* T.B.I. G is the $m-p$ dimensional vector bundle $G^\perp$ over N whose fiber over $n \in N$ is the orthogonal complement $\perp G(N_n)$ to $G(N_n)$ in $M_{g(n)}$. The *second fundamental form of* G, $\mathrm{II}_G: G^\perp \to \mathrm{Hom}(T(N), T(N))$ is a vector bundle map defined as follows. Given $v \in \perp G(N_n)$ and $x, y \in N_n$ extend y to a vector field Y on N in some neighborhood of n, let ∇ be the covariant derivation on M and put

$$\langle \mathrm{II}_G(v)x, y\rangle_n = -\langle \nabla_{Tg(x)} G(Y), v\rangle_{g(n)}.$$

The definition is independent of the choice of Y.

G is *parallel along* g if $(\mathrm{trace})\cdot \mathrm{II}_G: G^\perp \to R$ vanishes identically. It was shown in [1] that this definition is a generalization to higher dimensional immersed submanifolds, of the classical notion of parallelism along a curve. The significant facts are the following.

Every unit vector field along a curve $g: N^1 = (a, b) \to M$ corresponds in a natural way to a T.B.I. along g. Under this correspondence, parallel vector fields are paired with parallel T.B.I.'s.

An immersion $g: N^p \to M^m$ is isometric if and only if its tangent map

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