THE CENTRAL LIMIT THEOREM FOR THE RANGE OF TRANSIENT RANDOM WALK

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Let $\{X_n, n \geqq 1\}$ be a sequence of independent identically distributed random variables which take values in the d-dimensional integer lattice E_d. The sequence S_n defined by $S_0 = 0$, $S_n = \sum_{k=1}^{n} X_k$ for $n \geqq 1$ is called a random walk. If the distribution of the X_n assigns mass $(2d)^{-1}$ to each of the $2d$ neighbors of the origin, it is called *simple* random walk. Let $p = P[S_1 \neq 0, S_2 \neq 0, \cdots]$; the random walk is called transient if p is positive and recurrent otherwise. The random walk may take place on a proper subgroup of E_d. In this case, the subgroup is isomorphic to some E_k, $k \leqq d$; if $k < d$, then we should consider the problem in k dimensions. With this understanding, random walks with summands having mean zero and finite variance are transient if and only if $d \geqq 3$ [3].

Let R_n denote the cardinality of the set $\{S_0, S_1, \cdots, S_n\}$; R_n is called the range of the random walk (up to time n). Dvoretzky and Erdös [1] proved for simple random walk with $d \geqq 2$ that $R_n/ER_n \rightarrow 1$ with probability one. In the course of their investigation, they obtained the estimate $\operatorname{Var} R_n = O(n)$ for $d \geqq 5$. Jain and Orey considered the range of strongly transient random walk in [2]. (Random walks with summands having mean zero and finite variance are strongly transient if and only if $d \geqq 5$.) They proved that if $p < 1$ and the random walk is strongly transient, then $\operatorname{Var} R_n \sim \sigma^2 n$ for some positive constant σ^2 and also that R_n obeys the central limit theorem. The case $p = 1$ is uninteresting since then $R_n = n + 1$ almost surely.

We shall consider these problems for general random walk in three and four dimensions. The bounds that Dvoretzky and Erdös obtained here were $\operatorname{Var} R_n = O(n \log n)$ if $d = 4$ and $\operatorname{Var} R_n = O(n^{3/2})$ if $d = 3$. We have improved these bounds to $O(n)$ and $O(n \log n)$ respectively. Furthermore, we have proved that there is a positive constant σ^2 (which may depend on the distribution of the summands) such that $\operatorname{Var} R_n \sim \sigma^2 n$ if $d = 4$, while $\operatorname{Var} R_n \sim \sigma^2 n \log n$ if $d = 3$. (The result for $d = 3$ is only for the case where the summands have mean zero and finite variance.) With the asymptotic behavior of the variance available, we are then able to prove the central limit theorem.

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