HERMITIAN BILINEAR FORMS WHICH ARE NOT SEMIBOUNDED

BY ALAN MCINTOSH

Communicated by Felix Browder, November 5, 1969

1. **Introduction.** Many results from the theory of closed semibounded hermitian bilinear forms in a Hilbert space can be extended so as to apply to forms which are not semibounded and indeed have no restriction placed on their numerical range. Some of these results are summarized in this note, while details can be found in [**5**]. Our main interest here is in questions which arise naturally when the forms are assumed to be hermitian but not necessarily semibounded. Some such questions are: Is every closed hermitian form J of the type $J_A[u, v]=(A^{1/2}u, A^{1/2*}v)$ with A selfadjoint? Does every symmetric operator B with equal deficiency indices have a selfadjoint extension A such that the form J_A is minimal among those closed hermitian forms whose associated operators extend B? If such a minimal extension exists, is it necessarily unique?

If our forms and operators are semibounded then each of these questions is known to have a positive answer (see [**4**, Chapter V]). We will show that in the general case the third question may have a negative answer, and that the second question can be answered in the affirmative when there is a gap in the essential spectrum of B. Complete answers to the first two questions are not known to the author.

We remark that one reason for wanting to know whether a closed form is of the type J_A is that results from interpolation and spectral theory can then be used to obtain information about the form from our knowledge of the operator A.

2. **Closed forms.** If X and Y are complex linear spaces then a function J from $X\times Y$ to the complex numbers is called a *bilinear form* (or more properly a sesquilinear form) if it is linear in the first variable and conjugate linear in the second. In the case when X and Y are Hilbert spaces and the bilinear form J is bounded (meaning $|J[u, v]| \leq\gamma\|u\|_X\|v\|_Y$ for some positive γ and all $u\in X$ and $v\in Y$), there exists a bounded linear operator $C:X\to Y$ satisfying $J[u, v]=(Cu, v)_Y$ for $u\in X$ and $v\in Y$. If this operator C is an isomorphism

AMS Subject Classifications. Primary 4710; Secondary 4615.

Key Words and Phrases. Hilbert space, closed bilinear forms, nonsemibounded hermitian bilinear forms, selfadjoint operators, Friedrichs extensions.