ON PASTING BALLS TO HANDLEBODIES

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Throughout this paper all spaces will be simplicial complexes and all maps will be piecewise linear. We shall denote the boundary, closure, and interior of a space X by $\text{bd}(X)$, $\text{cl}(X)$ and $\text{int}(X)$ respectively. Let X be a space and Y a connected subspace. Then we shall denote the natural map induced by inclusion from $\pi_1(Y)$ into $\pi_1(X)$ by $\pi_1(Y)\to\pi_1(X)$.

We shall say that a submanifold X of a manifold Y is *properly embedded* in Y if $X\cap\text{bd}(Y)=\text{bd}(X)$. A *handlebody* is a 3-manifold homeomorphic to the regular neighborhood of a compact 1-complex embedded in E^3. If T_n is a handlebody and l is a simple loop in $\text{bd}(T_n)$, we can attach a disk to T_n by identifying the boundary of the disk with l. We may attach a thickened disk or a ball in a similar way to T_n and obtain a 3-manifold. When we perform the operation above we shall say that we have *pasted a ball to T_n along l.* We shall denote the smallest normal subgroup of $\pi_1(T_n)$ containing $[l]$ by $N(l)$.

It is the purpose of this article to prove:

THEOREM. *Let T_n be a handlebody of genus n. Let l be a simple loop in $\text{bd}(T_n)$ such that $\pi_1(T_n)/N(l)$ is free on $n-1$ generators. Then the 3-manifold obtained by pasting a ball to T_n along l is a handlebody of genus $n-1$.*

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PROOF. It follows from a theorem of Whitehead (see [2, p. 167, Theorem N3]) that $[l]$ can be taken to be a generator of $\pi_1(T_n)$. Let T_n' be homeomorphic to T_n under a map $h: T_n\to T_n'$. Then we can paste T_n to T_n' along regular neighborhoods in $\text{bd}(T_n)$, $\text{bd}(T_n')$ of l and $h(l)$ respectively, to obtain a 3-manifold M.

It is a consequence of Van Kampen's Theorem that $\pi_1(M)$ is free on $2n-1$ generators. Now $\pi_1(\text{bd}(M))$ is not free, so $\pi_1(\text{bd}(M))\to\pi_1(M)$ is not one-one. It follows from the loop theorem [3] that there is a disk properly embedded in M such that $\text{bd}(\mathfrak{D})$ is not

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