ON PERIODIC SOLUTIONS OF NONLINEAR HYPERBOLIC EQUATIONS AND THE CALCULUS OF VARIATIONS

BY MELVYN S. BERGER

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Let G be a bounded domain in R^N with boundary ∂G. Then the system (for $p(x)$ a strictly positive $C'(\overline{G})$ function)

$$
\begin{aligned}
p(x)\, u_{tt} - \Delta u &= 0 \qquad (\text{in } G),\\
u/\partial G &= 0,
\end{aligned}
\tag{1}
$$

has a countably infinite number of distinct periodic solutions (i.e. "normal modes"). In this note we shall show that the same conclusion can be established for the nonlinear system

$$
\begin{aligned}
p(x)\, u_{tt} - \Delta u + f(x, u) &= 0,\\
u/\partial G &= 0,
\end{aligned}
\tag{2}
$$

under certain restrictions on the functions $f(x, u)$ and $p(x)$. (Throughout we assume $f(x, 0) \equiv 0$, so that $u(x, t) \equiv 0$ satisfies (2).) Furthermore similar results can be obtained for higher order systems in which the Laplace operator Δ is replaced by a strongly elliptic operator of order $2m$ and the boundary conditions are suitably altered (such systems occur in the theory of elastic vibrations).

Our proofs are based on approximating the system (2) by a Hamiltonian system of ordinary differential equations, as in [**4**]. The periodic solutions of the associated Hamiltonian systems are then investigated by the methods of the calculus of variations in the large, as studied by the author in [**1**]. Periodic solutions of the original system (2) are then obtained by taking limits. Previous mathematical studies of periodic solutions of (2) (e.g. [**2**], [**3**], [**5**]) have been primarily perturbation results and have not considered the totality of periodic solutions of (2).

1. **Preliminaries.** Let x denote a point in G and $W_{1,2}(G_T)$ denote the Sobolev space of functions $u(x, t)$, T-periodic in t, which are square integrable and possess square integrable derivatives over $G \times [0, T]$. By $\dot{W}_{1,2}(G_T)$ we denote the subspace of $W_{1,2}(G_T)$ consisting of functions which vanish on ∂G (in the generalized sense). $\dot{W}_{1,2}(G_T)$ is a Hilbert space with respect to the inner product