ON THE SOLUTIONS OF THE NONLINEAR EIGENVALUE PROBLEM $Lu+\lambda b(x)u=g(x, u)$

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In connection with a problem in nonlinear reactor statics, we consider eigenvalue problems of the general form:

$$Lu + \lambda b(x)u = g(x, u), \quad x \in D, \tag{1}$$

$$\beta(x)\partial u/\partial \nu + \alpha(x)u = 0, \qquad x \in \partial D. \tag{2}$$

Here we take $x=(x_1, x_2, \cdots, x_m)$ and

$$\left.\begin{aligned} &L\phi \equiv \sum_{i,j=1}^{m} \frac{\partial}{\partial x_i}\left[a_{ij}(x)\frac{\partial}{\partial x_j}\phi\right] - a_0(x)\phi, \quad a_{ij}(x) = a_{ji}(x), \\ &\sum_{i,j=1}^{m} a_{ij}(x)\xi_i\xi_j \geqq a^2\sum_{i=1}^{m}\xi_i,\ a^2 > 0,\ a_0(x) \geqq 0, b\,(x) > 0, \end{aligned}\right\} x \in D; \tag{3}$$

$$\left.\begin{aligned} &\frac{\partial\phi}{\partial\nu} \equiv \sum_{i,j=1}^{m} n_i(x)a_{ij}(x)\frac{\partial}{\partial x_j}\phi, \qquad \alpha(x)\beta(x) \geqq 0, \\ &\alpha(x) \not\equiv 0,\ \alpha(x) + \beta(x) > 0, \end{aligned}\right\} x \in \partial D.$$

In addition, the functions $a_{ij}(x)$ and their derivatives are continuous on $\overline{D}$; the boundary is piecewise smooth with exterior unit normal $n(x)=(n_1(x), n_2(x), \cdots, n_m(x))$ for $x\in\partial D$. $g(x, u)$ is an analytic function of u.

The following lemma, which is established in substance by A. Hammerstein [1], is taken without proof here.

LEMMA. *Let the implicit equation,*

$$\sum_{m=2}^{\infty} L_{m0}\epsilon^m + \sum_{m=0}^{\infty}\epsilon^m\sum_{l=1}^{\infty}\mu^l L_{ml} = 0 \tag{4}$$

involving the small parameters ϵ and μ, be such that the coefficients L_{01} and L_{20} are nonzero. Then there are exactly two solutions of (4), *given by* $\epsilon(\mu) = \pm(L_{01}\mu/L_{20})^{1/2}[1+w(\mu)]$, *where $w(\mu)\to 0$ as $\mu\to 0$.*

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