AXIOM A+NO CYCLES$\Rightarrow\zeta_f(t)$ RATIONAL

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Throughout, $f: M\to M$ is a smooth diffeomorphism of a compact C^∞ manifold without boundary.

Let N_i denote the number of fixed points of f^i. Then

DEFINITION. $\zeta_f(t) = \exp\sum_{i=1}^{\infty} N_i t^i/i$ (as a formal power series in t). This definition, due to Artin-Mazur [1], is inspired by Weil's zeta function for a variety defined over a finite field [6]. For the connection of Weil's zeta function with the Riemann zeta function, see [3].

Recall the following definitions from *Differentiable dynamical systems* [4].

DEFINITION. $x\in M$ is *nonwandering* if for every neighborhood U of x, there is an $n>0$ such that $f^n(U)\cap U\neq\varnothing$. $\Omega(f)=\Omega$ is the set of nonwandering points of f. Ω is closed.

DEFINITION. f satisfies Axiom A if $T_\Omega(M)$ has a continuous splitting $T_\Omega(M)=E^s+E^u$, invariant under Tf, such that there exist positive constants c, λ, $\lambda<1$ satisfying the inequalities

$$\|Tf^n v\| \leqq c\lambda^n\|v\| \quad \text{if } n>0 \text{ and } v\in E^s,$$

$$\|Tf^n v\| \geqq c\lambda^{-n}\|v\| \quad \text{if } n>0 \text{ and } v\in E^u.$$

Furthermore, it is assumed that the periodic points of f are dense in Ω.

If f satisfies Axiom A, then $\Omega=\Omega_1\cup\cdots\cup\Omega_k$ where Ω_i is invariant under f and $f|\Omega_i$ is topologically transitive. Define the relation $\geqq$ by $\Omega_i\geqq\Omega_j$ if $W^u(\Omega_i)\cap W^s(\Omega_j)\neq\varnothing$. Here $W^u(\Omega_i)$ is the set of points tending toward Ω_i under negative iteration; $W^s(\Omega;\ )$ is the set of points tending toward Ω_j under iteration.

DEFINITION. If f satisfies Axiom A and the relation $\geqq$ defined above is a partial ordering, then f is said to have the *No Cycle Property*.

The purpose of this paper is to prove the following:

THEOREM. *If f satisfies Axiom* A *and the No Cycle Property, then $\zeta_f(t)$ is rational.*

The basic idea of the proof is due to Williams [7]. As a preliminary,

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