A NEW DISCRETE ANALOG OF THE LEGENDRE POLYNOMIALS

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1. **Introduction.** We introduce here a finite set of polynomials orthogonal over N distinct points of $[-1, 1]$ which are a very close analog of the Legendre polynomials. This set appears to have most of the properties of the Legendre (and ultraspheric) polynomials, yet they are not Fejér "generalized Legendre polynomials" (Szegö [4, §6.5]). These polynomials converge like $1/N^2$ to the Legendre polynomials, in contrast to the Hahn polynomials ($\alpha=\beta=0$) which converge like $1/N$. (See Karlin and McGregor [2], Levit [3].) In every respect, they appear to be a superior analog of the Legendre polynomials than the Hahn polynomials (sometimes called the Gram or Chebyshev polynomials of discrete least squares).

2. **The inner product.** The Hahn polynomials arise naturally from equidistant point sets. In approximation theory, useful point sets are the zeros of the Chebyshev polynomials of the first and second kind ($T_n(x)$ and $U_n(x)$). Let $\phi=\pi/(N+1)$, and $\psi=\frac{1}{2}\phi$. Let $t_i=\cos(i\phi)$, and $w_i=\sin(i\phi)$, $i=1, 2, \cdots, N$. Let $[\ ,\ ]_N$ be the inner product defined by $[f, g]_N=2\tan\psi\sum_{i=1}^{N} w_i f(t_i)g(t_i)$, and let $(f, g)=\int_{-1}^{1} f(t)g(t)dt$.

By means of the identity

$$\sum_{k=1}^{N} \sin k\alpha\phi = \cot\alpha\psi, \quad \alpha = 1, 3, 5, \cdots,$$
$$= 0, \quad \alpha = 0, 2, 4, \cdots$$

one can show $[1, 1]_N=2$, and that for fixed n, $N\geqq n\geqq 1$, $[t^n, t^n]_N$ monotonically increases (with N) to (t^n, t^n), and that $[t^n, t^n]_N=(t^n, t^n)+O(1/N^2)$.

If we let $q_k(t; N)$, $k=0, 1, \cdots, N-1$ be the monic form of the orthogonal polynomials of $[\ ,\ ]_N$, then, taking account of the symmetry of the inner product, the polynomials are given by the recurrence,

$$q_0 \equiv 1, \qquad q_1 \equiv t, \qquad q_{n+1} = tq_n - \beta_n(N)q_{n-1}, \qquad n = 1, 2, \cdots, N-2,$$

where $\beta_n(N)=[q_n, q_n]_N/[q_{n-1}, q_{n-1}]_N$. The polynomials $q_n(t; N)$ are

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