SURFACES OF VERTICAL ORDER 3 ARE TAME

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Communicated by Orville Harrold, July 17, 1969

We define a 2-sphere S in E^3 to have *vertical order n* if each vertical line intersects S in no more than n points. The main result in this paper is the following

THEOREM 1. *If S is a 2-sphere in E^3 having vertical order 3, then S is tame.*

This is the best theorem possible in the sense that examples are known of wild 2-spheres in E^3 having vertical order 4 [**5**]. In Theorem 2 to follow we generalize Theorem 1 to compact 2-manifolds in E^3.

Previous work concerned with the nature of the intersection of vertical lines with a 2-sphere in E^3 has been done by Bing [**1**, Theorem 7.3]; [**3**].

PROOF OF THEOREM 1. The vertical line in E^3 containing the point x is denoted by L_x, and we refer to the bounded component of E^3-S as Int S. If $x\in \text{Int } S$ it is easy to see that $L_x\cap S$ consists of two points. In this case the point with largest third coordinate is denoted by U_x and the other by V_x. We let $U=\{U_x \mid x\in \text{Int} S\}$ and $V=\{V_x\mid x\in \text{Int} S\}$, and we note that U and V are both open subsets of S. A bicollar can be constructed for a neighborhood of each point of $U\cup V$ using short vertical intervals. Thus S is locally tame at each point of $U\cup V$ [**2**].

Let $R=S-(U\cup V)$. The proof that S is tame is completed by showing that R is a tame simple closed curve, since a 2-sphere that is locally tame modulo a tame simple closed curve is known to be tame [**4**].

It will follow that R is a simple closed curve once we show that each of U and V is connected and that each point $p\in R$ is arcwise accessible from both U and V [**7**, p. 233]. Let θ be an arc in Int S $\cup\{p\}$ such that p is an endpoint of θ. We now show that the vertical projection σ of θ into $U\cup\{p\}$ is continuous. To accomplish this we take a sequence $\{x_i\}$ of points in θ converging to x_0 and we prove that the sequence $\{\sigma(x_i)\}$ converges to $\sigma(x_0)$. Let L_i $(i=0, 1, 2, \cdots)$ be the vertical interval from x_i to $\sigma(x_i)$ (if $x_i=p$, then L_i is degenerate),

AMS Subject Classifications. Primary 5705; Secondary 5478.

Key Words and Phrases. Tame 2-spheres, tame surfaces, surfaces in E^3.

[1] This research was partially supported by the National Science Foundation under GP-8454.