THE DENSEST LATTICE PACKING OF TETRAHEDRA

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The problem of finding the densest packing of tetrahedra was first suggested by Hilbert [3, p. 319]. Minkowski [4] attempted to find the densest lattice packing of tetrahedra, but his result is invalid due to the incorrect assumption that the difference body of a regular tetrahedron was a regular octahedron. A lower bound for the maximum density of such a packing has been given by Groemer [1] as 18/49. The purpose of this paper is to announce the proof that 18/49 is in fact the maximum possible density.

We shall use the term *convex body* to mean any compact, convex set in three dimensions with nonempty interior, and *lattice* to mean the collection of all points (vectors) $mA+nB+pC$, where m, n, p range over all integers and A, B, C are three fixed linearly independent vectors. If J is a convex body and Λ is a lattice such that, when x and y are distinct points of Λ, the bodies $x+J$ and $y+J$ have no interior points in common, then the collection of bodies $\Lambda+J = \{x+J: x\in\Lambda\}$ is said to form a *lattice packing*. If $\Delta(\Lambda) = |\det(A, B, C)|$, and $\mathrm{Vol}(J)$ represents the volume of J, then the *density* of the packing is defined to be $\mathrm{Vol}(J)/\Delta(\Lambda)$. Minkowski [4] showed that $\Lambda+J$ is a lattice packing if and only if there are no points of Λ other than the origin in the interior of the *difference body* $J-J = \{x-y: x, y\in J\}$. When the latter condition holds, Λ is said to be *admissible* for the difference body. A lattice is *critical* for a difference body if it is admissible and if no other admissible lattice has a smaller determinant. It follows that the problem of finding the densest lattice packing for a given convex body J is equivalent to that of finding a critical lattice for $J-J$. The following lemmas are also from Minkowski's paper.

LEMMA 1. *If* $\{A, B, C\}$ *is a basis for the lattice* Λ, *if* A, B, C *are on the boundary of the difference body* R, *and if none of the lattice points* $A\pm B$, $A\pm C$, $B\pm C$, $A\pm B\pm C$, $2A\pm B\pm C$, $A\pm 2B\pm C$, $A\pm B\pm 2C$ *is interior to* R, *then* Λ *is admissible for* R.

LEMMA 2. *If* Λ *is a critical lattice for the difference body* R, *then* Λ *has a basis* $\{A, B, C\}$ *such that* A, B, C, *and either*

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