MORSE THEORY ON BANACH MANIFOLDS

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S. Smale has conjectured, in an unpublished paper, that the Morse Theory on Hilbert mainfolds due to Palais and Smale [1], [4] can be extended to Banach manifolds. Under a different definition of nondegeneracy of critical points we have been able to make this extension. The result also extends Morse theory on Hilbert manifolds to a wider class of functions. I wish to thank R. Palais for several helpful suggestions.

Let f be a real-valued C^1 function on a C^1 Banach manifold X. A critical point x of f is said to be weakly nondegenerate if there exists a neighborhood U of x and a hyperbolic linear isomorphism L_x: $T_x(X) \to T_x(X)$ such that in the coordinate system of U, $df_{x+v}(L_x v) > 0$ for all $x+v$ in U, $v \neq 0$. Then $T_x(X)$ splits into the direct sum of two invariant subspaces of L_x, $T_x(X) \cong T_x(X)_+ \oplus T_x(X)_-$ such that the spectrum of L_x on $T_x(X)_+$ lies in the right half plane and the spectrum of L_x on $T_x(X)_-$ lies in the left half plane. The index of f at x is defined to be dim $T_x(X)_-$, and this term is well defined. A nondegenerate critical point of a function on a Hilbert manifold is weakly nondegenerate.

THEOREM 1. *Let f be a C^2 function on a C^2 paracompact manifold X without boundary modeled on a separable Banach space B. We assume that B has C^2 partitions of unity and a metric which is C^2 away from* 0. *If, in addition,*

(a) *f satisfies condition* (C) *of Palais and Smale with respect to a complete Finsler metric on X, and*

(b) *$q > q'$ are not critical values, and all the critical points in $f^{-1}((q, q'))$ are weakly nondegenerate of finite index,*

then there exists a homeomorphism θ: $f^{-1}[q, -\infty) \cong f^{-1}[q', -\infty) \cup h_i$ where a handle h_i of index q_i is added for each one of the finite number of critical points $x_i \in f^{-1}((q, q'))$ of index q_i.

REMARK. In the case of an infinite index, a similar result holds, provided that

$$df_{x+v}(L_x v) > \alpha(\|v\|_B) \qquad \text{for} \qquad 0 \neq v \in T_x(M)_- \cap U$$

where α is a continuous function from $R^+ \to R^+$.