UNITARY INVARIANTS FOR COMPACT OPERATORS

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We describe in this note how the "boundary representation" technique introduced in [1] leads to a complete classification of compact operators on Hilbert spaces to unitary equivalence (Theorem 3), in terms of a sequence of invariants related to (and generalizing) the numerical range. These invariants are, we feel, vastly simpler than one might have anticipated in so general a situation. Full details will appear in a forthcoming sequel to [1].

1. **Boundary representations for spaces of compact operators.** Let $LC(\mathfrak{H})$ (resp. $L(\mathfrak{H})$) denote the C^*-algebra of all compact (resp. bounded) operators on a Hilbert space $\mathfrak{H}$, which may be finite-dimensional. The following theorem implies, in the terminology of [1], that the identity representation of $LC(\mathfrak{H})$ is a boundary representation for every irreducible linear subspace of $LC(\mathfrak{H})$ (we call a set of operators irreducible if it commutes with no nontrivial self-adjoint projections).

THEOREM 1. *Let $\mathcal{S}$ be an irreducible subset of $LC(\mathfrak{H})$, and let ϕ be a completely positive linear map of $LC(\mathfrak{H})$ into $L(\mathfrak{H})$ such that $\|\phi\| \leqq 1$ and $\phi(T) = T$ for every T in $\mathcal{S}$. Then ϕ is the identity map.*

This result is surprising inasmuch as $\mathcal{S}$ can be a very small subset of $LC(\mathfrak{H})$ *a priori*. For example, $\mathcal{S}$ may consist of a single irreducible compact operator. We shall not give the proof of Theorem 1 here, except to say that it is an application of the following.

LEMMA. *Let $\mathcal{S}$ and ϕ satisfy the hypothesis of Theorem 1. Then there is a faithful, completely positive, idempotent linear map $\psi: L(\mathfrak{H}) \rightarrow L(\mathfrak{H})$ such that $\|\psi\| \leqq 1$, and whose compact fixed points coincide with the fixed points of ϕ.*

2. **The matrix range of an operator.** Let T be a Hilbert space operator, and let $C^*(T)$ denote the C^*-algebra generated by T and the identity. It is well known that, as ϕ runs over the state space of $C^*(T)$, the complex numbers $\phi(T)$ fill out the closure of the numerical

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