ORDER ALGEBRAS

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A partially ordered set P in which every pair of elements has a greatest lower bound is a semigroup, with $pq = p \wedge q$, and hence is naturally associated with a semigroup algebra $Z[P]$ over the integers. For finite P Solomon has given [3] a marvelously ingenious construction of an analogous sort of algebra even when P is not a semilattice and so cannot be made into a semigroup. Semigroup algebras and Solomon's "Möbius algebras" have applications in combinatorial problems involving the underlying orders.

Now in a recent study [2] of valuations and Euler characteristics on lattices Rota introduced an ostensibly quite different sort of algebra he called a "valuation ring" which, rather surprisingly, plays a role like that of a semigroup algebra. More surprising, in view of their entirely different genesis and description, is that Rota's valuation ring can be shown to include Solomon's Möbius algebra as a special case.

Rota's construction, when used to associate such an algebra to a partial order P (which is only one outgrowth of his inquiry), leads in stages through several different structures. The results implicitly provide a recursive procedure for computing products in the valuation ring $V(P)$, but give no direct formula. Solomon, on the other hand, defined his Möbius algebra by giving an explicit, if rather complicated, formula to express products of elements of P as linear combinations of P-elements. The purpose of this note is to determine from Rota's construction an explicit formula for products in $V(P)$ which depends only on the order structure of P. This will show at once that Rota's construction includes Solomon's, and it can be recast in a particularly simple form that clarifies further consequences and applications.

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