# ON AXIOMS FOR $B^*$-ALGEBRAS

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Let $B$ be a complex Banach algebra with an involution $x \to x^*$. Let $H$ denote the set of selfadjoint (s.a.) elements of $B$ and $W$ the subset of $H$ consisting of all $h \in H$ whose spectrum is entirely real. As in [3] we denote the spectral radius of $x \in B$ by $\nu(x)$. We prove the following result.

THEOREM. *Suppose that there exists $c > 0$ where $\nu(h) \geqq c\|h\|$ for all $h \in H$. Then $W$ is closed in $B$.*

This theorem has consequences for the theory of $B^*$-algebras. Shirali and Ford [4] have recently shown that $B$ is symmetric if $W = H$. Combining this and Lemma 2.6 of [6] with our theorem, we obtain the following result.

COROLLARY 1. *$B$ is a $B^*$-algebra in an equivalent norm if and only if $W$ is dense in $H$ and, for some $c > 0$, $\nu(h) \geqq c\|h\|$ for all $h \in H$.*

As usual $x \in B$ is said to be normal if $xx^* = x^*x$. Let $N$ denote the set of normal elements of $B$. Berkson [1] and Glickfeld [2] have shown (in case $B$ has an identity) that $B$ is a $B^*$-algebra in the given norm if $\|x^*x\| = \|x^*\|\|x\|$ for all $x \in N$. We obtain an analogous result for equivalence to a $B^*$-algebra.

COROLLARY 2. *$B$ is a $B^*$-algebra in an equivalent norm if and only if, for some $c > 0$, the set of $x \in N$ for which $\|x^*x\| \geqq c\|x^*\|\|x\|$ is dense in $N$ and contains $H$.*

We turn to the proof of our theorem. Let $B_1$ be the algebra obtained by adjoining an identity 1 to $B$ and defining, as usual, $\|\lambda + x\| = |\lambda| + \|x\|$ and $(\lambda + x)^* = \bar{\lambda} + x^*$ where $\lambda$ is complex and $x \in B$. We show that there exists $b > 0$ such that $\nu(y) \geqq b\|y\|$ for all $y$ s.a. in $B_1$. For suppose otherwise. Then there exists a sequence $\{\lambda_n + h_n\}$, with $\lambda_n$ real and $h_n \in H$, such that $|\lambda_n| + \|h_n\| = 1$ and $\nu(\lambda_n + h_n) \to 0$. By

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