LOCALIZED SOLUTIONS OF NONLINEAR WAVE EQUATIONS

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We consider complex valued solutions ϕ of nonlinear wave equations of the form

$$(1) \qquad \Box\phi = \phi_{tt} - \Delta\phi = -\phi v(|\phi|^2)$$

where v is the derivative of a positive definite potential V. That is

$$(2) \qquad \frac{dV(a)}{da} = v(a) \quad \text{and} \quad V(a) \geqq 0 \qquad \text{with } V(a) = 0 \quad \text{iff } a = 0.$$

We suppose $v(0) = m^2 > 0$.

A solution ϕ with finite energy is called localized if there is an $\epsilon > 0$ such that

$$(3) \qquad \sup_x |\phi(x, t)| = M(t) > \epsilon$$

whenever ϕ exists.

THEOREM. *If, for some* a_0

$$(4) \qquad V(a_0) < m^2 a_0$$

then equation (1) *has localized solutions.*

The proof is based on the conservation of *energy* $\mathcal{E}$ and *charge* $\mathcal{Q}$

$$(5) \qquad \mathcal{E} = \int \left\{ |\phi_t|^2 + \sum_{i=1}^{N} |\phi_{x_i}|^2 + V(|\phi|^2) \right\} dx,$$

$$(6) \qquad \mathcal{Q} = \operatorname{Im} \int (\phi_t \bar{\phi}) dx.$$

Suppose that $|\phi|^2 < \epsilon$. Then, from (2)

$$(7) \qquad V(|\phi|^2) > (m^2 - \delta)|\phi|^2$$

where δ tends to zero if ϵ does.

By the Schwartz inequality and (7) we easily deduce that

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