MATCHING THEOREMS FOR COMBINATORIAL GEOMETRIES

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Communicated by Gian-Carlo Rota, July 18, 1969

1. **Introduction.** Let $G(S)$ and $G(T)$ be combinatorial geometries of finite rank on sets S and T, respectively, and let $R \subseteq S \times T$ be a binary relation between the points of $G(S)$ and $G(T)$. By a *matching* from $G(S)$ into $G(T)$, we understand a one-one function f from an independent set $A \subseteq S$ onto an independent set $B \subseteq T$ with $(a, f(a)) \in R$ for all $a \in A$. In this note, we present a characterization of matchings of maximum cardinality, a max-min theorem, and a number of related results. In the case when $G(S)$ and $G(T)$ are both free geometries, Theorems 1 and 2 reduce to "the Hungarian method" as introduced by Egerváry and Kuhn [1], and to the König-Egerváry theorem, respectively. Corollary 2 for the case when $G(S)$ is a free geometry and $G(T)$ arbitrary was first discovered by Rado [6] (see also Crapo-Rota [2]). When both $G(S)$ and $G(T)$ are free geometries, Corollary 2 reduces to the well-known SDR theorem.

2. **Terminology.** For an arbitrary geometry $G(S)$, the closure operator will be denoted by J and the rank function by r. $(G(S), G(T), R)$ shall denote the system of the two geometries together with R, and $R(S') = \{y \mid \text{ there is some } x \in S' \text{ with } (x, y) \in R\}$ for $S' \subseteq S$. Let (A, B, f) denote a matching from A onto B. $M = \{(a, f(a)), a \in A\}$ is called the *edge set* of the matching (A, B, f), and we adopt the convention $M = (A, B, f)$. The common cardinality of A, B, M is called the *size* $\nu(M)$ of the matching. A *support* of $(G(S), G(T), R)$ is a pair (C, D) of closed sets, where $C \subseteq S$, $D \subseteq T$, such that $(c, d) \in R$ implies at least one of $c \in C$, $d \in D$ holds. The *order* λ of a support (C, D) is defined as $\lambda(C, D) = r(C) + r(D)$. Finally, an *augmenting* chain with

AMS Subject Classifications. Primary 0504, 0535, 0540; Secondary 0460, 0630.

Key Words and Phrases. Combinatorial geometry, matroid, binary relation between geometries, matching, Hungarian method, augmenting chain, König-Egerváry theorem, marriage theorem, transversal pregeometry.

[1] Research supported by the United States Air Force Grant AFOSR-68-1406.

[2] Research supported in part by National Science Foundation Grant No. GU-2059.