A NILPOTENT LIE ALGEBRA WITH NILPOTENT AUTOMORPHISM GROUP

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Recent work of Stein, Knapp, Koranyi and others has been concerned with nilpotent Lie groups which admit expanding automorphisms, that is, semisimple automorphisms whose eigenvalues are all greater than one in absolute value (cf. [1], [3], [5]). It was an open question whether all nilpotent Lie groups admit expansions.

We first present a result due to Louis Auslander (oral communication) which establishes the existence of a class of Lie groups which do admit expanding automorphisms (§1). We then present an example of a nine dimensional Lie group which does not have an expanding automorphism (§2). In our work it is more convenient to use Lie algebra language than Lie group language. As is well known, for connected simply connected nilpotent Lie groups the choice of group or algebra language is a matter of taste.

I wish to take this opportunity to thank Louis Auslander for suggesting this problem to me.

1. Let us begin by recalling some definitions. Details may be found in [2, Chapter V] or in [6, Chapter 5]. Henceforth algebra will mean algebra over the reals.

We say that $\mathfrak{L}$ is a *free Lie of algebra rank r* if there exist r elements $X_1, \cdots, X_r \in \mathfrak{L}$ which generate $\mathfrak{L}$ qua algebra and which enjoy the following universal mapping property: any function from the set $\{X_1, \cdots, X_r\}$ to any algebra $\mathcal{L}$ extends to a unique algebra homomorphism $\mathfrak{L} \rightarrow \mathcal{L}$.

Define the ideals $\mathfrak{L}^i$, $i=1, 2, \cdots$ of $\mathfrak{L}$ as follows:

$$\mathfrak{L}^1 = \mathfrak{L}, \qquad \mathfrak{L}^{i+1} = [\mathfrak{L}^i, \mathfrak{L}].$$

An ideal $\mathcal{I}$ of $\mathfrak{L}$ is *homogeneous* if the vector space $\mathcal{I}$ is isomorphic to the direct sum of $\mathcal{I} \cap \mathfrak{L}^i / \mathcal{I} \cap \mathfrak{L}^{i+1}$, $i=1, 2, \cdots$. We shall say that the Lie algebra $\mathcal{L}$ is homogeneous if $\mathcal{L}$ is isomorphic to $\mathfrak{L}/\mathcal{I}$ with $\mathfrak{L}$ free and $\mathcal{I}$ homogeneous.

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