MEASURE ALGEBRAS AND FUNCTIONS OF BOUNDED VARIATION ON IDEMPOTENT SEMIGROUPS[1]

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Let S be an abelian idempotent semigroup. Let T be a semigroup of semicharacters on S containing the identity semicharacter. A semicharacter on a semigroup S is a nonzero, bounded, complex valued function on S which is a semigroup homomorphism. A semicharacter on an idempotent semigroup is an idempotent function, and hence can assume only the values zero and one. We define $A_f = \{s \in S \mid f(s) = 1\}$ and $J_f = \{s \in S \mid f(s) = 0\}$ for each $f \in T$, and we denote by A the Boolean algebra of subsets of S generated by the sets $J_f (f \in T)$. If $X = \{f_1, \cdots, f_n\}$ is a finite subset of T, $\sigma \in T_n$ (T_n denotes the Boolean algebra of all n-tuples of zeros and ones), we define

$$B(X, \sigma) = \{ \bigcap_{\sigma(i)=1} A_{f_i} \} \cap \{ \bigcap_{\sigma(i)=0} J_{f_i} \}. \tag{1}$$

Clearly, A consists of finite unions of sets of the form (1). If F is a function on T, X and σ are as above, we define an operator L by

$$L(X, \sigma)F = \sum_{\tau \in T_n} \mu(\sigma, \tau) F\Big(\prod_{\tau \geq \sigma} f_i^{\tau(i)} \Big), \tag{2}$$

where

$$\begin{aligned} \mu(\sigma, \tau) &= (-1)^{|\tau| - |\sigma|} & \tau &\geqq \sigma, \\ &= 0 & &\text{otherwise}, \end{aligned}$$

is the Möbius function for T_n [3]. Here $|\sigma|$ denotes the number of ones in the n-tuple σ. We call F a function of bounded variation on T if

$$\sup_X \sum_{\sigma \in T_n} | L(X, \sigma)F | < \infty, \tag{3}$$

where the supremum is taken over finite subsets X of T. The norm of F is the number defined by (3). Finally, we say that F is positive definite if

$$L(X, \sigma)F \geqq 0 \tag{4}$$

[1] These results were obtained in the author's doctoral dissertation written at the University of Utah under the direction of Professor Joseph L. Taylor.