FACTORIZATION OF OPERATOR VALUED ENTIRE FUNCTIONS

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Communicated by Irving Glicksberg, June 23, 1969

Let $W(z)$ be a complex valued entire function of exponential type with nonnegative values on the real axis. We call $W(z)$ *factorable* if

$$W(z) = A^*(z)A(z)$$

where $A(z)$ is an entire function whose restriction to the upper half-plane is an outer function. Here $A^*(z) = \bar{A}(\bar{z})$. Recall that an outer function in the upper half-plane is a function of the form

$$f(z) = C \exp\left(\frac{1}{\pi i}\int_{-\infty}^{+\infty} \frac{1+tz}{t-z} \frac{\log k(t)}{1+t^2} dt\right), \qquad y > 0,$$

where C is a constant of absolute value 1, $k(t) \geqq 0$ a.e. on $(-\infty, \infty)$, and $(1+t^2)^{-1} \log k(t) \in L^1(-\infty, \infty)$. Of necessity, $k(x) = \lim |f(x+iy)|$ a.e. where the limit is taken as y decreases to zero. Therefore the restriction of an entire function $A(z)$ to the upper half-plane is an outer function if and only if $(1+t^2)^{-1} \log|A(t)| \in L^1(-\infty, \infty)$ and

$$\log|A(z)| = \frac{y}{\pi}\int_{-\infty}^{+\infty} \frac{\log|A(t)|}{(t-x)^2+y^2} dt, \qquad y > 0.$$

The following facts are available from the classical theory of entire functions:

(1°) for $W(z)$ to be factorable, it is necessary and sufficient that

$$\int_{-\infty}^{+\infty} \frac{\log^+ W(x)}{1+x^2} dx < \infty,$$

(2°) if $W(z)$ is factorable, the factor $A(z)$ is determined to within a multiplicative constant of absolute value 1,

(3°) if $W(z)$ is factorable, say $W(z) = A^*(z)A(z)$ as above, and if $W(z)$ is of exponential type τ, then $\exp(-\frac{1}{2}i\tau z)A(z)$ is of exponential type $\frac{1}{2}\tau$. See [2, p. 125], [3, p. 34], and [4, p. 437], where some original sources are cited. The purpose of this note is to communicate extensions of these results to operator valued entire functions.

[1] Research supported by NSF Grant GP 8981.